



**THE DEVELOPMENT OF PRESERVICE TEACHERS' CONTENT KNOWLEDGE FOR
TEACHING EARLY ALGEBRA**

by

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ABSTRACT

The purpose of this study was to understand the development of preservice teachers' knowledge for teaching early algebra as a result of an early algebra course and teaching practicum.

Preservice teachers enter teacher education with a diversity of school experiences of learning algebra which usually involves a high degree of procedural understanding. This study argues the importance of preservice teachers having the experience and opportunity to develop both conceptual and procedural understanding of the mathematics they will teach.

The research was based on a case study, using qualitative methodologies and framed within an interpretive paradigm. It included a group of third year preservice teachers studying for a Bachelor of Education degree in the General Education and Training (GET) band. The early algebra course, known as Maths 2, was designed to develop knowledge for teaching early algebra and to build mathematical proficiency through participation in a professional learning community. The design and content of the course were guided by Ball's mathematical knowledge for teaching (MKfT) model and the choice of early algebra as functional thinking aligned with the goals of the Revised National Curriculum for Mathematics (RNCS) and the more recent Curriculum Assessment Policy Statement (CAPS) in the mathematics content area: patterns, functions and algebra.

The preservice teachers' development of knowledge for teaching early algebra was identified through their manifestations of knowledge for teaching early algebra. These manifestations were illustrated by preservice teachers' verbal and written responses from lesson reflections, questionnaires as well as video recordings of selected lessons. Focus group interviews were used to investigate the role of the early algebra course (Maths 2) in developing preservice teachers' knowledge for teaching early algebra.

The findings indicate that preservice teachers developed both common content knowledge (CCK) and specialised content knowledge (SCK) for teaching early algebra. Their responses indicated a growing awareness of the development of their mathematical knowledge for teaching through their own experiences of a richer and connected algebra and through guided support and reflection in the process of learning and teaching.

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LIST OF ACRONYMS

ANA	Annual National Assessment
CAPS	Curriculum and Assessment Policy Statement
CCK	Common Content Knowledge
CK	Curricular Knowledge
DBE	Department of Basic Education
DHET	Department of Higher Education
DoE	Department of Education
EA	Early Algebra
FP	Foundation Phase
GET	General Education and Training
HK	Horizon Knowledge
IP	Intermediate Phase
KCC	Knowledge of Content and Curriculum
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
MfT	Mathematics for Teaching
MKfT	Mathematical Knowledge for Teaching
MKiT	Mathematics Knowledge in Teaching
PCK	Pedagogical Content Knowledge
PST	Preservice teachers
PUFM	Profound Understanding of Mathematics
RNCS	Revised National Curriculum Statement
SCK	Specialised Content Knowledge
SMK	Subject Matter Knowledge
SP	Senior Phase

CHAPTER 1. INTRODUCTION AND OVERVIEW

1.1. Introduction and background

This study links two important topics within the realm of education in South Africa: teacher knowledge and mathematics. Regular media coverage of teacher inefficiency and poor mathematics results nationally both contribute to the overall perception that learners in this country are losing out. There is no doubt that the political legacy of the past has much to account for in this regard but there is an urgent need for a multi-faceted approach to finding solutions to the crisis in education. Teacher education can be an effective tool in helping build the knowledge and skills of inservice teachers and to develop the knowledge and skills of preservice teachers. This study is located in this context and seeks to understand the development of preservice teachers' knowledge for teaching early algebra. The focus on algebra arises from the weak Grade 12 results in mathematics and the exciting international research into the development of algebraic reasoning in the early grades.

The following section (1.2) gives an overview of the context of education and the related issues in South Africa. It begins by looking at the mathematics results from Grades 1 - 12, and then considers the results from recent studies of teacher knowledge in South Africa. This is followed by a rationale for a focus on algebra teaching and learning in schools with some discussion about the related curriculum expectations. It concludes with an overview of the current government policies and plans for the teacher education and support of teachers in the future. Sections 1.3 and 1.4 explain the rationale and purpose for the study, followed by a summary of the theoretical framework that guided the research (1.5). There is a brief explanation of the research design and methodology (1.6) and limitations of the study (1.7) with some discussion of the significance of this research for the future (1.8). The chapter concludes with an outline of the thesis (1.9).

1.2. Background to the issues that guided this study

Jansen (2011) identifies five reasons for the low productivity of the school system in South Africa: lack of systematic routines and rituals; knowledge problem; bureaucratic and administrative ineptitude; lack of accountability and lack of capacity and expertise. The knowledge problem is of particular interest to those involved in education and more especially teacher education. While there are numerous

inservice teacher training programs provided by national and provincial education departments, they are overly generic and do not address the lack of content knowledge in key subjects such as mathematics and science. The government's decision to prioritise mathematics and science is obviously driven by the technological and scarce skills need of the country. However without the political control of schools and the lack of expert knowledge on the part of teachers, there is little chance for technological advancement and job creation in the current system.

1.2.1. Mathematics results: Grade 1 - 12

Interestingly South Africa spent more than 20% of the national budget for 2012/13 on education which is a great deal more than most other African countries, but with moderate success. The 2012 Annual National Assessments¹ (ANAs) reveal a disturbing state of affairs in schools. Given the 50% pass mark required to pass the ANA at a specific grade, the average marks of 13% in mathematics at Grade 9, 27% at Grade 6 and 41% at the Grade 3 level are serious cause for concern. A summary of the results are as follows (Department of Basic Education, 2012a):

Table 1.1: National average percentage marks for Mathematics in 2011 and 2012

GRADE	MATHEMATICS 2012	MATHEMATICS 2011
1	68	63
2	57	55
3	41	28
4	37	28
5	30	28
6	27	30
9	13	(not assessed)

The picture is a little better at the other end of the spectrum in which National Senior Certificate² – Grade 12 mathematics results showed that 54% of those who wrote the subject mathematics were able to pass and achieve above 30% - the commonly accepted pass mark - which is an improvement from 2011 (DBE, 2012b) as Table 1.2 overleaf shows:

¹ Annual National Assessments (ANAs) are standardised national assessments for languages and mathematics in the intermediate phase (grades 4 – 6) and in literacy and numeracy for the foundation phase (grades 1 – 3).
² National Senior Certificate (NSC) is the school-leaving certificate in South Africa.

Table 1.2: Candidates' performance at 30% and above in maths and maths literacy 2011-2012

Subjects	2011 (total no. of matrics = 496 090)			2012 (total no. of matrics = 511 152)		
	Total wrote	Achieved at 30% & above	% achieved	Total wrote	Achieved at 30% & above	% achieved
Mathematics	224 635	104 033	46.3	225 874	121 970	54.0
Maths Literacy	275 380	236 548	85.9	291 341	254 611	87.4

However it is disconcerting that of the total number of candidates (511 152) who wrote the National Senior Certificate for Grade 12, less than 50% wrote mathematics and just over 50% of those managed to achieve 30% and above. This means that approximately 24% of learners wrote and passed mathematics at the Grade 12 level in the National Senior Certificate which has dire consequences for university applications and for production of graduates in key growth areas.

1.2.2. Teacher knowledge

As the national mathematics assessments continue to show low learner achievements, there is also growing evidence to suggest that many South African teachers do not have the content knowledge (CK) and pedagogical content knowledge (PCK) required to teach mathematics effectively. This means that the political problems of the past continue to impact on another generation of learners with the result that a large number of schools still do not have teachers who can teach mathematics at the appropriate level. The Carnoy et al. (2008) report which looked at factors contributing to low levels of mathematics achievement in South African primary schools have found some evidence to suggest that high quality mathematics teaching is positively related to learner achievement. Although the sample of 40 schools is relatively small, the study results support claims that improvements in learner achievement are the result of improved teaching by those who know more about the subject and how to teach it. There are also some tenuous results indicating that PCK is also related to the quality of the preservice teacher education received. The results of the Carnoy et al. study highlight teacher knowledge as an important factor in learner achievement, and suggest that teachers need to know more about the subject as well as how to teach it to improve learner achievement.

Carnoy, Chisholm and Chibisa (2012) conducted research with South African and Botswana teachers and learners and found some evidence to suggest that teachers with good mathematical knowledge taught mathematics more effectively, and were likely to spend more time teaching mathematics. These teachers in turn produced better learner results overall. Few teachers in either country were rated best quality due to their low levels of mathematical pedagogical content knowledge. Carnoy et al. (2012) suggested that more emphasis was needed in inservice teacher education to develop teachers' mathematical content and pedagogical content knowledge as well as guidance in teaching the required national curriculum standards. They mentioned that improvements in the overall quality of practising teachers had proved to be a challenging task which means the role of preservice teacher education becomes imperative. It is essential for teacher education institutions to make sure that prospective teachers are better prepared in terms of their understanding of mathematics and their knowledge to teach mathematics in schools. There is no doubt that this country and our learners need teachers with strong mathematical knowledge for teaching particularly in the area of algebra, since algebra is a foundation for success in learners' future mathematical studies.

1.2.3. Algebraic thinking

The introduction of compulsory mathematics subjects: mathematics or mathematical literacy³ at the National Senior Certificate level has boosted the number of learners being exposed to some level of mathematics. However the low numbers of learners electing to take mathematics at senior grades means that vital courses at universities related to engineering, technology, commerce, medicine and science continue to be under-subscribed. The content of the mathematical literacy course does not provide learners with an adequate level of mathematics to be able to cope with the scientific, technological and building related subjects and limits educational and economic opportunities for many learners. Often, learners are not taking mathematics at the Grade 10 to 12 levels because of the cognitive demands of the subject and the poor mathematical foundations they have gained in the lower grades. One of the pillars for success in mathematics at the high school level is the ability to reason algebraically. This is however not being developed sufficiently in

³ Mathematical literacy (ML) provides learners with an awareness and understanding of the role that mathematics plays in the modern world. Mathematical literacy is a subject driven by life-related applications of mathematics. It enables learners to develop the ability and confidence to think numerically and spatially in order to interpret and critically analyse everyday situations and to solve problems (DoE, 2003).

the lower grades thus limiting learners' access to further mathematics related courses. Lack of knowledge and proficiency in algebra then becomes seen as a gatekeeper to opportunity and a mechanism of inequity (Kaput, 1998).

Success in future mathematics depends largely on algebra knowledge and skills which develop throughout school, especially in high school. The move from arithmetic thinking to algebraic thinking is notoriously difficult and cognitively demanding for learners and can have the effect of restricting educational and economic opportunities. Usiskin (2004) highlights the importance of algebra as the language of generalisation to enable a person to describe relationships and solve everyday problems. The introduction of pre-algebra at the early high school level is one approach to the teaching and learning of algebra but there is considerable research now available that shows children thinking algebraically from an even earlier age. The inclusion of algebra thinking in the early grades, or early algebra (EA) as it is also called, gives learners the opportunity to experience and develop conceptual understanding of algebra from the outset. Early algebra can help to prepare learners for more increasingly complex mathematics, to cultivate habits of mind that attend to the "deeper underlying structure of mathematics" and to improve their algebra understanding (Blanton & Kaput, 2005a: 412). It is not about introducing algebra early but rather about helping learners to "learn to reason algebraically" and to "begin to acquire a symbolic, algebraic language for expressing and justifying their ideas" (Blanton, 2008: 7). The emergence of early algebra research has emphasised the opportunity to develop children's mathematical reasoning from early in their school career and cultivate habits of mind for understanding mathematics that will prevail through to the higher grades and other areas of mathematics (Kaput & Blanton, 2008). Hunter (2010) suggests that the development of early algebra thinking for young children needs to go beyond making conjectures and include experiences that involve mathematical reasoning with justifications and generalisations. Children need to have access to a notational system which helps them to express their generalisations such as the use of number sentences.

1.2.4. Curriculum guidelines

The *Revised National Curriculum Statements*⁴ for primary school mathematics (Department of Education: 2002) and the more recent *Curriculum and Assessment*

⁴ Revised National Curriculum Statement (RNCS) is a revision of the National Curriculum Statement (C2005).

*Policy Statement*⁵ (DBE, 2011a) recognise the importance of developing algebraic principles from an early stage through the inclusion of one of the main content area: Patterns, Functions and Algebra. It identifies the general content focus as: “Algebra is the language for investigating and communicating most of mathematics and can be extended to the study of functions and other relationships between variables” (DBE, 2011a: 9). There is a deliberate shift from seeing algebra as a set of rules and procedures to a more conceptual understanding of what it means to do algebra. This shift emerges from substantive research with young children in relation to algebraic thinking. We now know that children are capable of reasoning algebraically at a younger age which can develop parallel to their arithmetic development (Lins & Kaput, 2004). However a problem arises when teachers, who have previous experience and understanding of what constitutes algebra, are required to shift their understanding and teaching approach to include a more extended notion of algebra and algebra activity. This is particularly pertinent to preservice teachers who have had only one experience of algebra during their own school years and have yet to experience something different.

1.2.5. Teacher education

Kallaway & Sieborger (2012) highlight the difficulty of teacher education in preparing teachers who are competent and confident to teach the major academic disciplines given the financial constraints operating at most university education departments in the country. They also draw attention to the importance of teacher knowledge both in terms of its impact on teacher practice and the motivation of learners. The latest plan to address the challenges in teacher education is the *Integrated Strategic Planning for Teacher Education and Development in South Africa: 2011 – 2025* (Department of Higher Education and Training, 2011a) a long term, ambitious set of recommendations to turn education around. The plan which was compiled with input from a range of stakeholders from across teacher education, provides a detailed set of outcomes and outputs, and strategies to enable the implementation of the plan and an accompanying planning framework with timelines, costs and targets.

The Department of Higher Education and Training has taken additional measures to address the critical challenges facing education by acknowledging the “poor content

⁵ Curriculum and Assessment Policy (CAPS) is an amendment to the National Curriculum Statement (NCS) Grades R-12.

and conceptual knowledge found among teachers” (DHET, 2011b: 6) and as mentioned previously by revising the *Minimum Requirements for Teacher Education Qualifications*. While recognising that teaching is a complex activity, the new policy emphasises that a good foundation in knowledge needs to underpin the skills required for teaching. There are five types of learning which are associated with the acquisition, integration and application of knowledge for teaching purposes, namely: disciplinary, pedagogical, practical, fundamental and situational.

Disciplinary learning or *subject matter knowledge* has two components, the study of education and its foundations as well as the study of “specific *specialised subject matter* that is relevant to the academic discipline underpinning the teaching subject” (p. 8). Pedagogical learning incorporates *general pedagogical knowledge* (knowledge of learners, learning, curriculum, teaching and assessment) and *specialised pedagogical content knowledge* (knowledge of how to represent concepts, methods and rules to create appropriate learning opportunities for diverse learners and how to evaluate progress). Practical knowledge involves learning in and from practice and includes the *study of practice* through case studies and videos to help theorise practice and form a basis for learning in practice. It is an important condition for the development of tacit knowledge which is essential for learning to teach. Fundamental knowledge refers to general *literacies*, being competent in a second official language, the ability to use information and communication technologies and the acquisition of academic literacy that forms the foundation of effective learning in higher education. The last type of learning is situational and is knowledge about different learning situations, contexts and environments of education (poverty, HIV and AID, diversity, inclusivity, etc.) and includes current educational policy, political and organisational contexts.

Each type of learning is relevant to the development of teachers’ knowledge for teaching and to produce confident and competent teachers to teach mathematics. According to the DHET (2011b) *Minimum Requirements for Teacher Education Qualifications* policy, it is important for newly qualified teachers to have sound subject knowledge, to know how to teach their subject and to select, sequence and pace content depending on the needs of the subject and the learners. It is also important to have knowledge of the school curriculum and to be able to unpack its specialised content which includes using available resources to plan and design lessons. While there are other basic competences that a beginner teacher needs,

the group of competencies selected above link specifically to the focus of this research on the development of content knowledge for teaching early algebra. Although the preservice teachers are not yet beginner teachers, the under-graduate courses that they take at university are intended to build their competencies towards becoming a teacher.

1.3. Rationale for the study

Through my work in preservice teacher education over the past fifteen years, I have had the opportunity to meet many prospective primary school teachers who have a lack of confidence and anxiety about mathematics, often due to a weak conceptual understanding of the subject. Through the teacher education process, they are encouraged to develop a deep, flexible and inter-connected understanding of the concepts they will teach. This entails developing content knowledge of mathematics as well as specialised content knowledge of mathematics that is particular to the work of teaching (Ball, Thames & Phelps, 2008). It is crucial and most important that teachers know and are able to “use the mathematics required inside the work of teaching” (p. 403). Kajander and Jarvis (2009) work with Canadian elementary teachers and also support a focus on the unique mathematical knowledge for teaching. They suggest that primary school teachers need to develop “specialised mathematical knowledge for effective classroom teaching” which is a critical component to “support improved learner learning and outcomes” (p. 14).

This study emerged from the context of my work in teacher education and from a desire to educate preservice teachers that are proficient to teach mathematics. The research focused on mathematics in the primary school to acknowledge the importance of a good foundation in mathematics and involved a group of preservice primary school teachers who had chosen to become mathematics specialist teachers. They had an interest in mathematics teaching and learning and had performed well in their previous mathematics education courses. They had elected to take a third year mathematics education course, known as Maths 2, which focused on the development of algebraic thinking in the primary school. The Maths 2 course is designed to help preservice teachers to make connections between knowledge of early algebra as well as knowledge of teaching early algebra and to contribute to their effectiveness in the mathematics classroom. While there continues to be debate on the relationship between teacher knowledge and learner achievement as well as teacher qualification and learner performance, there is no

doubt that identifying, developing and deepening teachers' content knowledge for teaching is a priority for both policy makers and mathematics educators everywhere.

1.4. Purpose of the study

The purpose of this study is to understand the development of preservice teachers' content knowledge for teaching early algebra in the primary school. It links current research on teacher knowledge and early algebra and follows the development of preservice teachers' content knowledge as they engage with a Maths 2 course and teaching practicum. It recognises that preservice teachers come with a diversity of experiences of learning algebra from school which involves a high degree of procedural understanding. However it is important in teacher education that preservice teachers have the opportunity to develop both conceptual and procedural understanding of the mathematics they will teach. They need to understand that teacher knowledge is continually changing and developing, is not static and grows through interactions with mathematical learning in the classroom environment through engagement with learners, and through preservice professional experiences (Fennema & Franke, 1992).

The term subject content knowledge generally refers to subject matter knowledge (SMK), pedagogical content knowledge (PCK) and curriculum knowledge (CK) (Shulman, 1986). Knowledge for teaching, in this study, refers to subject matter knowledge only and is comprised of common content knowledge and specialised content knowledge as outlined in the mathematical knowledge for teaching model (Ball, Hill & Bass, 2005). Further details of the theoretical model follow in section 1.5 and Chapter 2. Early algebra (EA) represents a different approach to algebra teaching which highlights learner's ability to think and reason algebraically in the primary grades, and helps learners to cultivate habits of mind for understanding mathematics that prevail through to the higher grades and other areas of mathematics (Blanton, 2008).

1.4.1. Research questions

The main research question and sub-questions which guided this study were as follows:

Research question:

How does preservice teachers' knowledge for teaching early algebra develop as a result of an early algebra course and teaching practicum?

Research sub-questions:

1. What manifestations of knowledge for teaching early algebra do preservice teachers illustrate in their verbal and written feedback taken from an early algebra course and teaching practicum?
2. What manifestations of knowledge for teaching early algebra do preservice teachers demonstrate in their teaching of an early algebra lesson during teaching practicum?
3. What aspects of the early algebra course and teaching practicum contribute to the development of knowledge for teaching early algebra?

1.5. Theoretical framework

Two constructs guide the theoretical framework for this study into the development of preservice teacher knowledge for teaching early algebra, namely knowledge for teaching and early algebra. There are many different perspectives of knowledge for teaching mathematics which each attend to both the mathematics and the work of teaching. The Fennema and Franke (1992) model examines the nature of knowledge for teaching mathematics and identifies four elements: knowledge of content, knowledge of pedagogy, knowledge of learners' cognition and teacher beliefs. Davis and Simmt (2006) offer a different perspective which focuses on mathematics and the nature of mathematical knowledge embodied and enacted in teaching while others attend to the classification of situations in which mathematical knowledge surfaces in teaching (Rowland et al., 2009).

The theoretical approach to knowledge for teaching used in this study is based on the mathematical knowledge for teaching (MKfT) model which is a framework used to categorise and assess the knowledge needed for teaching mathematics (Ball et al., 2008). It is the result of two decades of research into the practice of teaching by identifying teaching tasks examining the mathematics needed to complete these

tasks and using this to analyse the knowledge needed for teaching. The MKfT model is a practice-based theory which focuses on the kind of professional knowledge of mathematics which is different from that demanded by other mathematically intensive occupations. It moves beyond the limiting boundaries of knowledge to include skills, reasoning, habits of mind and sensitivities needed to teach mathematics effectively (Ball et al., 2008).

MKfT elaborates the use of the Shulman’s (1986) subject matter knowledge, pedagogical content knowledge and curriculum knowledge categories, organising and defining them in a different way. There are six elements to MKfT: common content knowledge (CCK), specialised content knowledge (SCK), horizon content knowledge (HCK), knowledge of content and teaching (KCT), knowledge of content and learners (KCS), and knowledge of content and the curriculum (KCC). The organisation of the domains of knowledge is indicated in Figure 1.1 below, followed by definitions of each of the domain. The conceptual history and development of the model is discussed in more detail in Chapter 2.

1.5.1. Theoretical Framework for Mathematical knowledge for teaching (Ball et al., 2008):

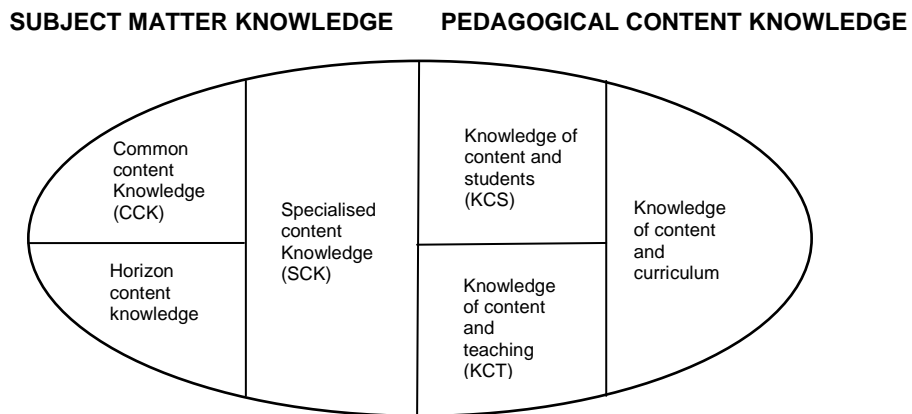


Figure 1.1: Domains of mathematical knowledge for teaching

Definition of Framework Terms (Ball et al., 2008: 399-402)

Common content knowledge (CCK): the mathematical knowledge and skill used in settings other than teaching. It involves correctly solving mathematical problems. Teachers need to know the work they must teach, recognise incorrect answers and use mathematical terms and notation correctly.

Specialised content knowledge (SCK): the mathematical knowledge and skill unique to teaching. Teachers have to be able to do a kind of mathematical work that others do not. It involves unique mathematical understanding and reasoning and requires knowledge beyond that being taught to learners. Understanding different interpretations of mathematical problems and solutions and helping learners to make sense of their work and that of others is crucial. It involves making features of particular content visible to and learnable by learners and explaining how mathematical language is used. Teachers must choose, make and use mathematical representations that are effective and help students to explain and justify their mathematical ideas.

Horizon content knowledge (HCK): a provisional inclusion in the model and is an awareness of how mathematical topics are relayed over the span of mathematics included in the curriculum. This is about teachers' understanding what the mathematics looks like in grades either above or below the grade that they are instructing.

Knowledge of content and teaching (KCT): combines knowing about teaching and knowing about mathematics. Teachers have to have a mathematical knowledge of the design of instruction. They must be able to sequence particular content for instruction and select and use effective instructional models. They have to identify what different methods and procedures to use to make the mathematics salient and usable by learners. It involves making decisions about what learner contributions to pursue and which to ignore or save for a later time.

Knowledge of content and students (KCS): knowledge that combines knowing about learners and knowing about mathematics. It involves anticipating what learners are likely to think and what they will find confusing as well as possible difficulties they may experience. It also involves recognising errors and identifying most likely errors learners make, interpreting learner responses and developing learner justifications. Teachers have to know how to select lessons and assessment tasks that are interesting and motivating for learners.

Knowledge of content and curriculum (KCC): linked to Shulman's curricular knowledge and is currently placed within pedagogical content knowledge and supported by later work of Shulman and colleagues.

This study concentrates on preservice teachers' knowledge for teaching early algebra and looks specifically at their espoused and enacted understanding of common content knowledge (CCK) and specialised content knowledge (SCK) for teaching early algebra as a result of the Maths 2 course and teaching practicum.

1.5.2. Early Algebra

The second construct of the theoretical framework for this research largely draws from the work of Kaput (2008: 11 - 14) into algebra and algebraic thinking and is based on many years of work in the field of early algebra. A detailed discussion of the development of early algebra is outlined in more detail in Chapter 2. Kaput identifies two core aspects of algebra as:

- Core Aspect A - Algebra as systematically symbolising generalisations of regularities and constraints i.e. using symbols to generalise; and
- Core Aspect B - Algebra as syntactically guided reasoning and actions on generalisations expressed in conventional symbol systems i.e. acting on symbols, and following rules.

These core aspects run through each of three algebraic strands:

- Strand 1: Generalised arithmetic - Algebra as the study of structures and systems in arithmetic and quantitative reasoning
- Strand 2: Functional thinking - Algebra as the study of functions, relations and joint variation; and
- Strand 3: Modelling - Algebra as the application of a cluster of modelling languages both in and out of mathematics.

The focus on early algebra in this study involves using both symbols to generalise (Core A) and acting on symbols (Core B) in the context of functional thinking (Strand 2) and modelling (Strand 3). Functional thinking and modelling are linked to the Revised National Curriculum Statement (DoE, 2002) and Curriculum Assessment Policy Statement (DBE, 2011a) mathematics content area: Patterns, Functions and Algebra. The focus on functional thinking does not mean that generalising from arithmetic and quantitative reasoning is not equally important. However, this study centres on functional thinking as it provides a context for developing ways of thinking algebraically within activities by creating opportunities for learners to study change and analyse relationships, to notice structure, to generalise, to problem solve, to model, to justify, to prove and to predict (Kieran, 2004).

The recent Curriculum Assessment Policy Statement (DBE, 2011a, 2011b) documents indicate that at both the Foundation Phase (Grades 1 - 3) and Intermediate Phase (Grades 4 - 6), learners need to work with numbers and geometric patterns. Teachers in the Foundation Phase (FP) should use physical objects, drawings and symbolic forms to copy, extend, describe and create patterns focusing on the logic of patterns and lay the basis for developing algebraic thinking skills (DBE, 2011a). The Intermediate Phase (IP) teachers need to extend numeric and geometric patterns with a special focus on the relationships between terms in a sequence and between the number of terms (its place in the sequence) and the terms itself. It includes developing the concepts of variables, relationships and functions to describe the rules generating the patterns. It has a particular focus on the use of different, yet equivalent, representations to describe problems or relationships by means of flow diagrams, tables, number sentences or verbally (DBE, 2011b).

There are many reasons given in literature for using patterns as generalising tasks in the mathematics classroom. Barbosa, Vale and Palhares (2009: 2) highlight the usefulness of patterns to help build a “more positive and meaningful image of mathematics” as well as develop crucial skills related to “problem solving and algebraic thinking”. Mason, Graham and Johnston-Wilder (2005) suggest that manipulating familiar objects can inspire confidence and is the beginning of structure. The structure eventually emerges in the form of a generalisation or expression. Trying out a case involving familiar objects and moving from the simple case towards analysing a collection of particular cases helps build early generalising. It requires a supportive environment in which everything said is taken as a conjecture to be tested out in experience. Specialising and generalising are not just activities to make tasks interesting, they lie at the heart of mathematical thinking, and familiarity with generalising is much more likely to support algebraic thinking than learning how to manipulate algebraic symbols too early. Samson and Schafer (2007) indicate that results from research literature highlight the benefits from working with pattern activities which include opportunities to engage with algebraic thinking processes which are a precursor to formal algebra. Pattern activities also create the chance to problem solve through the search for patterns and help develop a critical habit of mind (Cuoco, Goldenburg & Mark, 1996).

Pattern activities provide a special opportunity for teachers to develop a particular kind of generality in learners' thinking, i.e. an immersion in the "culture of algebra" (Lins & Kaput, 2004). It provides them with a set of experiences that enables them to see mathematics – sometimes called the science of patterns - as something they can make sense of, and to provide them with the habits of mind that will support the use of the specific mathematical tools that they will encounter when they teach algebra in the primary school (Schoenfeld, 2008).

Pattern activities provide a rich and accessible entry point into early algebra as functional thinking and a useful way to develop algebraic thinking in the early grades (Anthony & Hunter, 2008). They have the potential to be a powerful tool for developing "an understanding of the dependent relations among quantities that underlie mathematical functions" as well as a "concrete and transparent way for young students to begin to grapple with the notions of abstraction and generalizations" (Moss & Beatty, 2006:193). Pattern tasks which lead to generalisation are important in making the transition from arithmetic to algebraic thinking and provide a useful introduction to the concept of variable and future work with symbols (Zazkis & Liljedahl, 2002; Stalo et al., 2006;). Generalising towards the idea of a function means to recognise regularity through elementary patterning. It involves ideas of change including linearity and representation through tables, flow diagrams/function machines and graphs.

1.6. Research design

The focus of this study, namely to understand the development of preservice teachers' knowledge for teaching early algebra lends itself to qualitative research. Qualitative research focuses on exploring and understanding the "meaning individuals or groups ascribe to a social or human problem" (Creswell, 2009: 4). This qualitative study works within an interpretivist paradigm in that it is interested in understanding the meanings that preservice teachers themselves ascribe to their experiences as a result of the early algebra course, known as Maths 2, and teaching practicum, and what this tells us about the development of their knowledge for teaching early algebra. The study explores these meanings through the verbal and written responses of the preservice teachers taken from their written reflections, questionnaires and focus group interviews and through their actions in the classroom in the video recorded lessons. The study supports the qualitative paradigm as it does not conceive of people as separate from the world but rather as

constructors and creators of their own reality (Bryman, 2008). The Maths 2 course and the teaching practicum give preservice teachers the opportunity to construct their own interpretations of knowledge for teaching early algebra and the research methods and analysis help the researcher to interpret these meanings to better understand the development of knowledge for teaching early algebra. This is a case study of a group of preservice teachers studying for a Bachelor of Education - Foundation Phase (FP) and Intermediate/Senior Phase (ISP) degree at a university in the Western Cape, South Africa.

1.6.1. Sample

The group was an opportunistic sample of twenty six preservice primary school teachers, in the third year of their degree and comprised of 9 FP preservice teachers (all female) and 17 ISP preservice teachers (4 male and 13 female). They had elected to take a mathematics specialisation course (Maths 2) which focused on the development of knowledge for teaching early algebra in the General Education and Training (GET) band. The Maths 2 course was designed for FP and ISP preservice teachers who wanted to become mathematics specialist teachers in the primary school.

1.6.2. Data collection methods

This research centred on understanding the development of knowledge for teaching early algebra and necessitated a variety of different methods to capture the depth and variety of the preservice teachers' thoughts and actions. Data collection methods included video lesson recordings, lesson reflections, questionnaires on the videos and focus group interviews conducted during and at the end of the course to provide different perspectives of the same phenomena in support of the case study methodology. The data collection methods also provided opportunity for the preservice teachers to reflect upon and develop their knowledge for teaching early algebra as they progressed through the Maths 2 course and teaching practicum experience.

They firstly prepared and taught an early algebra lesson based on the knowledge for teaching early algebra given in the Maths 2 course. Data was first collected from twenty six early algebra lessons which were videoed and later transcribed. Secondly, there were post-lesson interviews to discuss the lesson after which the preservice teachers used notes from these interviews to prepare lesson reflections.

The third set of data came from the questionnaires which the preservice teachers used to reflect on the video of their early algebra lesson. The questionnaire focused on their specialised knowledge for teaching early algebra and more specifically on some of the tasks related to the teaching of early algebra completed during the lesson. Lastly, two focus groups of preservice teachers were interviewed to reflect on the knowledge for teaching early algebra content covered in the course, the teaching methods of the course, and the teaching practicum. These four sets of data were used to understand the meanings that preservice teachers themselves ascribe to their experiences as a result of the Maths 2 course and the teaching practicum and what this tells us about the development of their knowledge for teaching early algebra.

1.6.3. Data analysis

The knowledge for teaching early algebra course focused specifically on the preservice teachers' common content knowledge (CCK) and specialised content knowledge (SCK) for teaching early algebra (Ball et al., 2008). The analysis process was threefold and started by looking across the verbal and written responses (reflections, questionnaires and interviews) of the preservice teachers for manifestations of their knowledge for teaching early algebra. The responses were then grouped into common content knowledge, specialised content knowledge and knowledge of early algebra (functional thinking) guided by the theoretical framework of the study. These were then analysed further to understand the nature of the development of knowledge for teaching early algebra.

The second part of the analysis involved looking for manifestations of knowledge for teaching early algebra as demonstrated in the video lesson recordings and focussed exclusively on specialised knowledge for teaching early algebra i.e. six tasks of teaching: defining, explanations, representations, working with learners' ideas, restructuring tasks, and questioning (Kazima, Pillay & Adler, 2008). The final part of the analysis looked at the focus group interviews to identify what aspects of the Maths 2 course contributed to the development of knowledge for teaching early algebra as indicated by the preservice teachers.

1.6.4. Measures of quality

This case study involved multiple data sources in the form of interviews, observations (videos), and documentation (reflections and questionnaires) to allow

for intensive examination of the development of knowledge for teaching early algebra and presents a holistic understanding of the case being studied (Bryman, 2008). Each source of data can be seen as a piece of the puzzle which contributed to the overall understanding of the development of teacher knowledge and early algebra. Where the findings from the data converged, there emerged a stronger and greater understanding of the preservice teacher knowledge for teaching early algebra (Baxter & Jack, 2008).

There were three measures of quality used to build the trustworthiness of this study. These are explained in detail in chapter 3: *transferability* which involved the use of detailed descriptions within the study; *credibility* (internal validity) through the provision of detailed description and explanation of the case study; and *dependability and confirmability* which was evident in the methodological rigour and explanation of biases of the study (Guba, 1981 in Rule & John, 2011: 107).

1.6.5. Ethical considerations

A letter of application for ethical clearance was prepared and permission granted from the University Ethics Committee to conduct the research with the preservice teachers and in schools. The purpose of the research study was explained at the beginning of the academic year to all of the preservice teachers and they were given the choice to participate or not. All preservice teachers agreed to take part in the research and were assured of confidentiality in all distribution of the findings.

The video recordings were conducted in schools and permission to access and video was negotiated between the Western Cape Education Department and the respective school principals. Letters of explanation and request for permission to conduct the research were given to the school principals and the parents of learners to explain the nature and purpose of the research and to give assurance of confidentiality of all information. Parents were given the option to withdraw their child from the lesson while filming took place.

1.7. Limitations of the study

There is an inevitable issue of subjectivity when conducting case study research and more especially in teacher education when the mathematics education researcher and mathematics educator is one and the same person. There is the obvious close relationship between the mathematics educator and the preservice teachers and

while there may be questions of interpretation, this research tries to give a detailed, rich description of the process of data collection, the Maths 2 course and teaching practicum and the data analysis. Each of the findings of the research questions starts with links to literature and theory, followed by illustrations taken from the verbal and written responses as well as excerpts from the videos of the early algebra lessons, and closes with interpretation and discussion of the illustrated findings.

This study does not attempt to generalise the observations made from the findings as they reflect the opinions and actions of individuals within a particular group. The data collection process does not try to circumvent the issue of bias by using a variety of data collection methods to support the findings but rather uses multiple sources to provide additional perspectives and to highlight the multi-faceted nature of knowledge for teaching early algebra (Rule & John, 2011). This case study foregrounds the mathematical knowledge for teaching framework to understand how knowledge for teaching early algebra develops through firstly, understanding the meanings that preservice teachers ascribe to their experiences of the Maths 2 course and teaching practicum and secondly, through their actions in the classroom.

This case has an instrumental rather than an intrinsic value in that it explores the broader issue of knowledge for teaching early algebra using a group of preservice teachers and makes use of the MKfT model to attend to both the mathematics and the work of teaching. It also engages dialogically between the theory and the case to reflect on the MKfT framework in the context of this study.

1.8. Significance of the study

This study is important because it helps us to understand knowledge for teaching by considering the connections a group of preservice teachers make from a mathematics education course. It is an opportunity to integrate knowledge of early algebra and knowledge for teaching early algebra and to understand what sense preservice teachers make of the knowledge and its application in the mathematics classrooms at the primary school level. The Maths 2 course was designed to integrate disciplinary learning, pedagogical learning and practical learning in a different way using alternative teaching strategies to help create a mathematical learning community of preservice teachers (DHET, 2011b). This study provides the

opportunity to interrogate and engage with this process and to inform future preservice mathematics education courses.

There is also opportunity to explore and engage with the field of early algebra both in terms of the content and literature and its application in the classroom. The study helps in understanding the nature of algebra thinking in the early grades and to recognise particular aspects of algebra content and how this can be transformed to build opportunities to explore, express and justify mathematical relationships. There is also the chance to attend to the teaching practices that help develop algebraic thinking through questioning learners, working with their ideas, helping learners to represent their ideas in different ways and identifying mathematical relationships (Blanton, 2008). The findings of this aspect of the research will help generate valuable input for continued professional development of preservice and inservice teachers.

The analysis of the videos, interviews and artefacts (reflections and questionnaires) using the mathematical knowledge for teaching model helps give insight into teacher knowledge from the perspective of practice. It attends to both the mathematics of the lesson as well as the work of teaching and provides evidence of the tasks of teaching and the issues and demands these tasks present for preservice teachers. This is useful in focussing the purpose of mathematics teacher education programmes.

Practical learning involves learning in and from practice which includes the study of practice so as to theorise practice and to learn in and from authentic classroom environments (DHET, 2011b). It is important for preservice teachers to have the experience of teaching an early algebra lesson to build tacit knowledge which is a vital part of learning to teach and which cannot be learned in books but through the practical experience of the classroom. This research is essential in understanding how preservice teachers begin to make sense of this process both in terms of their espoused and enacted knowledge of teaching early algebra.

There is also much to learn from the preservice teachers' responses in terms of the features of the Maths 2 course which help develop their knowledge for teaching early algebra. Their experiences and observations come from a year long exposure to mathematical content, mathematical pedagogical content of early algebra and

practical application. They are best situated to be able to analyse this process and to comment on the broader notion of teacher knowledge both as a process as well as a product (Barker, 2007).

1.9. Outline of the thesis

The first chapter provides an overview of the research problem and process and includes a context and rationale for the study. This is followed by some discussion of teacher knowledge, early algebra and the aims of the national curriculum for mathematics. There is an explanation of the purpose of the study and the research questions including some elaboration of the theoretical framework for knowledge for teaching early algebra. This is followed by a short overview of the research design and methodology, including some limitations of the study as well as the significance of the study towards research in preservice teacher education, knowledge for teaching and early algebra. The chapter concludes with an outline of the structure of the thesis.

The second chapter provides a literature review of perspectives of teacher knowledge and current research in early algebra. It starts with a general discussion of teacher knowledge and the seminal work of Shulman (1986) and links to the mathematical knowledge for teaching (MKfT) model (Ball et al., 2008). There is also some description of other perspectives on teacher knowledge which are currently being used in mathematics education research. The second part of the review focuses on early algebra, starting with definitions of algebra and linking to early algebra. There is an overview of different interpretations of early algebra from a variety of countries and research projects including research links to learning, teaching and the curriculum.

The third chapter deals with the methodology of the study. There is an explanation of the choice of paradigm and case study methodology including issues of credibility and reliability, followed by a description of the site and sample of the study. The purpose and content of the Maths 2 course is outlined as well as the data collection methods with links to the research questions. The chapter concludes with a detailed explanation of the process and method of data analysis.

The fourth chapter details the findings and interpretation of each of the research questions, starting with verbal and written responses taken from the post-lesson

reflections, questionnaires and the focus group interviews. The feedback on the development of content knowledge for teaching early algebra is reported in terms of both common content knowledge and specialised content knowledge for teaching and linked to literature from Chapter 2. There are five video recordings selected from the preservice teachers' early algebra lessons focussing specifically on the enacted specialised content knowledge for teaching. A summary of the findings on the contribution of the Maths 2 course to the development of knowledge for teaching includes interpretation of the findings and further links to literature.

The fifth and final chapter represents the discussion and conclusion of the research. There is an overview of the findings and an elaborated discussion of interesting aspects of the development of knowledge for teaching early algebra that were challenging for this group of preservice teachers. Elements of the Maths 2 course are unpacked in terms of their contribution to the development of knowledge for teaching and there is a critique of the role and efficacy of the MKfT model. The chapter concludes with a discussion of the implications of this research for teacher education, teacher knowledge and early algebra, including recommendations for future research.

CHAPTER 2. LITERATURE REVIEW

2.1. Introduction

This chapter presents an overview of the theory and literature which informed the theoretical framework for the study that aims to understand preservice teachers' development of knowledge for teaching early algebra as a result of the Maths 2 course and teaching practicum. Two different constructs are presented and discussed which relate to this framework: *teacher knowledge* and *early algebra*.

The two constructs form the organisation of the chapter and each is elaborated on in terms of theory and related empirical research. The first section (2.2) of this chapter deals with the construct of teacher knowledge and investigates the meaning, importance and limitations of several theoretical models for teachers' mathematical knowledge which have emerged over the past thirty years in different parts of the globe. The discussion starts with a brief outline of the seminal work of Shulman (1986, 1987), and its elaboration within the American and Australian contexts (Fennema and Franke, 1992; Ma, 1999; Chick et al., 2006; Ball et al., 2004, 2005, 2008). This is followed by a discussion of more recent work in mathematical knowledge in teaching emanating from England (Goulding & Petrou, 2008, 2011; Rowland et al., 2009; Watson & Barton, 2011) and South Africa (Adler & Davis, 2006; Adler & Pillay, 2007; Kazima et al., 2008). There is also linkage to other work on teacher knowledge in other parts of the world, where appropriate. The summary of empirical research on teacher knowledge touches on key aspects of knowledge for teaching which are relevant to the study; it does not attempt to present a complete review of the extensive field of teacher knowledge in mathematics.

The second section (2.3) of the chapter focuses on the construct: *early algebra*. It traces the roots of early algebra thinking and research, and provides a rationale for its importance within primary mathematics teaching and learning. This is followed by an overview of the research on early algebra as it relates to learners, teachers and the curriculum with a summary of key findings. The chapter concludes (2.4) with a summary of the two constructs and an elaboration of the theoretical framework of this study.

2.2. Teacher knowledge perspectives in mathematics education

Much has been written about the importance of teacher knowledge but there is little consensus on the content and structure of that knowledge, or agreement on a single accepted framework for describing teachers' mathematical knowledge in teaching (Clay & Fischer, 2010; Even & Tirosh, 2008). It has been the subject of intense writing and study over the past thirty years, led initially by the work of Shulman (1986, 1987). His research on identifying and describing categories of teacher knowledge and more particularly his work on teacher content knowledge (subject matter knowledge, pedagogical content knowledge and curricular knowledge) has been greatly influential and has led to much debate on the knowledge needed for teaching.

While there may be no agreement on a widely accepted model for teacher knowledge that can be used to fully describe knowledge for teaching, it is accepted that the work of Shulman and colleagues "initiated a new wave of thinking about teacher knowledge by suggesting that content should matter in teaching" (Goulding & Petrou, 2008: 1). Previous research recognises the role of subject matter knowledge and pedagogical knowledge but paid little attention to a special body of knowledge for teaching. Rather than seeing teacher education from the perspective of either content or pedagogy, Shulman (1987) proposes to consider the relationship between two knowledge bases as the intersection of content and pedagogy and introduced the notion of pedagogical content knowledge (PCK). It is the "special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (Shulman 1987: 8). The term PCK was the start of a new way of thinking about knowledge for teaching and broadened the scope of teacher knowledge to move beyond discipline content knowledge alone (Chick & Harris, 2007).

Ball et al. (2008: 390-391) identify three purposes to Shulman and colleagues' initial work:

- to reframe the study of teacher knowledge in ways that attend to the role of content in teaching
- to provide a comparative basis for examining more general characteristics of the knowledge that teachers use in practice
- to draw from these categories of teacher knowledge to inform the development of a National Board system for the certification of teachers

Shulman (1986, 1987: 20) was ultimately concerned about the limited notion of teacher competency which focused too much on “generic teaching behaviour”. He proposed a framework for analysing teachers’ knowledge that extended the description of teacher knowledge to include seven general and content-specific dimensions. These are:

General dimensions of teacher education

- General pedagogical knowledge
- Knowledge of learners and their characteristics
- Knowledge of educational context
- Knowledge of educational ends, purposes and values

Content-specific dimensions

- Subject matter content knowledge
- Curricular knowledge
- Pedagogical content knowledge

The content-specific dimensions or categories were described as the missing paradigm in research on teaching at the time and have been the focus of on-going research in education for the past 30 years (Petrou & Goulding, 2011). The categories have been explained as follows:

Subject matter content knowledge (SMK): “refers to the amount and the organisation of knowledge per se in the mind of the teacher” (Shulman, 1986: 9). This means that the teacher must not only understand *that* something is so but the teacher must further understand *why* it is so. It comprises the theories, concepts and principles and the approaches to generating and verifying ideas (Weiss et al., 2006: 2). This knowledge dimension has been further extended through the work of Grossman, Wilson and Shulman (1989) into three sub-categories: content knowledge, substantive knowledge and syntactic knowledge. Substantive knowledge concerns the organisation of key facts, theories, models and concepts of mathematics while syntactic knowledge deals with the processes by which theories and models are generated and established as valid (Petrou and Goulding, 2011).

Curricular knowledge (CK) is “represented by the full range of programs designed for the teaching of particular subjects and the topics at a given level” (Shulman,

1986: 10). It is the variety of instructional materials available in relation to those programs, and the characteristics that serve as both the indications and the contra-indications for the use of particular curriculum or materials in different circumstances. It is the understanding of how ideas are introduced, sequenced, and connected in instructional materials or courses (Weiss et al., 2006).

Pedagogical content knowledge (PCK) is the third kind of content-specific dimension of knowledge and goes beyond knowledge of subject matter to the “dimension of subject matter knowledge *for teaching*” which is the knowledge needed for teaching the subject. Shulman (1987: 8) defined PCK as:

....the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organised, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction.

PCK is knowledge particular to the work of teaching and includes “ways of representing and formulating the subject that make it comprehensible to others”, and “understanding what makes the learning of specific topics easy or difficult” (Shulman, 1986: 9). It is not just the knowledge of the subject or the knowledge of pedagogy needed for teaching but rather a blend of content and pedagogy into an understanding of how particular topics, problems, or issues are organised, represented, and adapted to the different interests and abilities of learners, and then presented for instruction. It is about the connection between content, pedagogy and learners (Baker and Chick, 2006). It includes the knowledge of the most useful forms of representations: “illustrations, examples, explanations and demonstrations, knowledge of learner conceptions and preconceptions” (Shulman, 1986: 9).

Shulman did not set out to compile a list of what teachers need to know in any particular subject but tried to develop a conceptual orientation and a set of analytical distinctions on the nature and types of knowledge needed for teaching a subject (Ball et al., 2008: 392). He accepted that one type of content knowledge cannot replace another. This is supported by the research of Ma (1999) who conducted a comparative study of the work of Chinese and American mathematics teachers. She concludes that pedagogical content knowledge (PCK) cannot replace limited subject matter knowledge (SMK), as teachers who do not have subject content understanding cannot ask thought-provoking questions and do not have the understanding of “fundamental mathematics that is deep, broad and thorough” (Ma

, 1999: 124). She articulates a framework to evaluate the depth and richness of teachers' content knowledge by examining their profound understanding of fundamental mathematics (PUFM) which is a deep knowledge of the mathematics at the level they are teaching. She defines PUFM as:

....more than a sound conceptual understanding of elementary mathematics - it is the awareness of the conceptual structure and basic attitudes of mathematics inherent in elementary mathematics and the ability to provide a foundation for the conceptual structure and instil those basic attitudes in students. (p. 120)

Ma's (1999) research indicates that Chinese teachers have higher levels of PUFM which equip them to teach more effectively, than their American counterparts. As Shulman notes in his forward to Ma's book, Chinese teachers may have studied far less mathematics than the American teachers, but what they know, they know "more profoundly, more flexibly, more adaptively" (Ma, 1999: xi). Moreover, Chinese teachers continuously refine their knowledge through "deliberations with their colleagues on the content of their lessons" and "conceptual knowledge for teaching is as much about pedagogy as it is about content" for these teachers (Ma, 1999: x-xi). It would appear that Ma's Chinese teachers exhibit highly structured "knowledge packages" which indicate characteristics of all three types of Shulman's knowledge and are developed through a "culture of discussion" (p. 113).

Many researchers over the years, including Ma, have examined the nature of Shulman's knowledge for teaching in more detail. The Fennema and Franke (1992) model of teacher knowledge expands and adapts Shulman's work to suggest knowledge needed for teaching is both interactive and dynamic in nature and centres on teacher knowledge as it occurs in the context of the classroom. They argue knowledge for teaching, as it occurs in the classroom, comprises four elements: *knowledge of content*, *knowledge of pedagogy*, *knowledge of learner cognitions* as well as *teachers' beliefs*. Teaching is recognised as a process in which new knowledge is created as a result of the interaction between the teacher's knowledge of content, knowledge of pedagogy and learner cognitions and teacher beliefs. In other words, teachers have to take complex mathematical knowledge and transform it for the teaching context so that learners can interact with the content and learn. This transformation process happens differently both for teachers and learners and is on-going, depending on the context.

Fennema and Franke (1992: 162) are interested in studying and defining more clearly the components of teacher knowledge and their interrelationships which help build a deeper understanding of teacher knowledge. They caution that measuring knowledge at a given time could ignore the complexity of “how knowledge changes as teachers participate in various experiences, both planned and unplanned”. They emphasise the point that knowledge needed in teaching is ‘*interactive and dynamic in nature*’ and challenge future research to develop a methodology that focuses on “understanding the interaction between the different theoretical knowledge domains” (Goulding & Petrou, 2008: 1).

Chick et al. (2006) in Australia have worked with teachers to unpack the concept of PCK in the context of primary schools. Their emphasis on PCK is not to negate the role of SMK but rather to better understand and define teachers’ knowledge for teaching. Their framework is an elaboration of the work of other researchers in the field of mathematics education including research in the area of representations, student thinking, texts and materials, teaching strategies, PUFM and their own research into teachers’ PCK. Through their large scale investigation of teachers’ PCK, they have designed a comprehensive framework to analyse teachers’ PCK comprising three parts: *Clearly PCK; Content Knowledge in a Pedagogical Context and Pedagogical Knowledge in a Content Context*.

Clearly PCK refers to knowledge of teaching strategies, student thinking, cognitive demands of tasks, representations, resources, curriculum, and purpose of content knowledge which are a blend of knowledge of content and pedagogy.

Content Knowledge in a Pedagogical Context is more specific to mathematical content knowledge in teaching: profound understanding of fundamental mathematics (PUFM), deconstructing content to key components, mathematical structure and connection, procedural knowledge and methods of solutions.

Pedagogical Knowledge in a Content Context is knowledge drawn from pedagogy and includes goals for learning, getting and maintaining student focus and classroom techniques.

They have used this framework and the sixteen sub-categories to analyse teachers’ PCK through written questionnaires and interviews and found clear differences between the PCK held by teachers. While some of the categories revealed teachers

had similar thinking, the framework helped also give richer and deeper understandings within categories. Teachers spoke in different ways about their work and learners, which was not always a consequence of their initial training or subsequent professional development but more connected to their beliefs about the nature of mathematics. The framework recognises the multi-faceted nature of teacher knowledge and is a useful tool in helping to unpack the different components of PCK as well as illuminate the similarities and differences between teachers and their PCK.

The framework does not try to describe the interconnection between SMK and PCK nor look at the development of knowledge for teaching. Rather, it is more concerned with providing descriptions about how teachers currently hold pedagogical content knowledge through their answers and descriptions.

Pedagogical content knowledge (PCK) appears to bridge subject matter content knowledge and the practice of teaching, but the bridge between knowledge and practice is not clearly understood nor has a single coherent theoretical framework been developed. Although the term PCK is used frequently and interpreted widely, and claims are made of what teachers need to know, there are few studies which empirically test whether there are “distinct bodies of identifiable content knowledge that matter for teaching” (Ball et al, 2008: 390). There is clearly a need to understand what teacher knowledge includes so researchers and teacher educators can begin to investigate and develop a more effective curriculum for educational purposes (Goulding & Petrou, 2008).

While models of teacher knowledge for teaching continue to be developed and refined, the research work of Ball and colleagues at the University of Michigan is an attempt to unravel and clarify in some detail the notions of SMK and PCK. It is a model of teacher knowledge that takes the work of Shulman and tries to develop a reliable and valid measure of mathematical knowledge for teaching and uses the measures to test theoretical definitions and models of teacher knowledge (Ball et al, 2005).

2.2.1. Mathematical Knowledge for Teaching (MKfT)

Over the past decade, Ball and her colleagues have worked on two key research projects, namely the Mathematics Teaching and Learning to Teach project and the

Learning Mathematics for Teaching project. Through this work, they have developed a model of teacher knowledge by examining of the actual work of mathematics teaching in primary schools. It is a practice-based theory of what they call “mathematical knowledge for teaching – a kind of professional knowledge of mathematics different from that demanded by other mathematically intensive occupations” e.g. engineering and carpentry, and constitutes the mathematical knowledge needed to carry out the work of teaching mathematics (Ball et al 2005; Ball et al, 2008).

Ball et al (2008) have incorporated Shulman’s subject matter content knowledge, pedagogical content knowledge and curricular knowledge into the umbrella term “mathematical knowledge for teaching”. Ball’s model of mathematical knowledge for teaching uses Shulman’s division between SMK and PCK and distinguishes three components of SMK namely: common content knowledge (CCK), specialised content knowledge (SCK) and horizon knowledge (HK). PCK is comprised of knowledge of content and teaching (KCT), knowledge of content and students (KCS) and knowledge of content and curriculum (KCC). A comparison between the two classifications of knowledge for teaching looks as follows:

Shulman	Ball
Subject Matter Knowledge (SMK)	Common Content Knowledge (CCK) Specialised Content Knowledge (SCK) Horizon Knowledge (HR)
Pedagogical Content Knowledge (PCK)	Knowledge of content and teaching (KCT) Knowledge of content and students (KCS) Knowledge of content and curriculum (KCC)
Curricular Knowledge (CK)	

Figure 2.1: Comparison of the Shulman and Ball models

The mathematical knowledge for teaching model proposed by Ball et al. (2008: 403) and presented in the figure 2.2 below is based on empirical evidence. The model was developed from analysing the work of teaching, identifying the teaching tasks involved and the knowledge needed to teach mathematics effectively. It is based on the premise that teachers need to know mathematics and know how to use mathematics in the work of teaching learners (Ball et al., 2008). It is a theory of

teacher knowledge “framed in relation to practice” and is composed of following domains of knowledge:

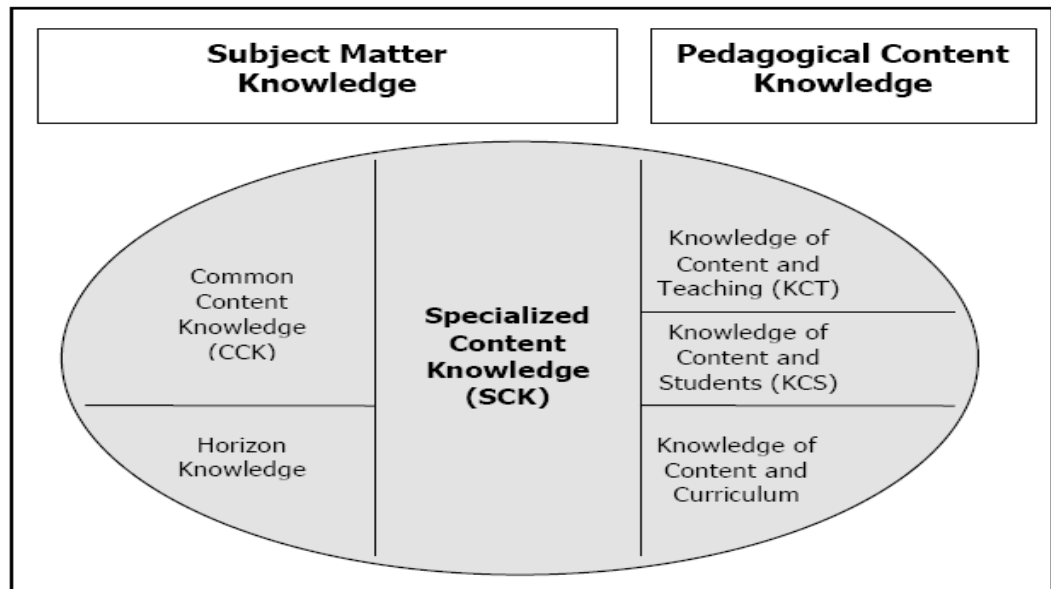


Figure 2.2: Model of mathematical knowledge for teaching

Ball and her colleagues are interested in identifying the recurrent tasks and problems of teaching mathematics and the mathematical knowledge, skills and sensibilities required to manage these tasks. It is an approach to developing theory about mathematical knowledge needed for teaching; it is not about what the teacher knows or might need to know but rather a description of knowledge used in teaching. They define mathematical knowledge for teaching as mathematical knowledge *entailed by teaching*, in other words, mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to learners (Ball et al, 2009: 97).

The model is comprised of subject matter knowledge and pedagogical content knowledge which is further sub-divided into common content knowledge (CCK), specialised content knowledge (SCK) and horizon knowledge as well as knowledge of content and teaching (KCT), knowledge of content and students (KCS) and knowledge of content and curriculum (KCC). These are explained as follows (Ball et al., 2008: 399 – 402):

CCK (Common content knowledge) is the mathematical knowledge and skill used in settings other than teaching. It involves correctly solving mathematics problems. Teachers need to know the work they must teach, recognise incorrect answers and

use mathematical terms and notation correctly. Knowledge of the school mathematics curriculum is crucial in designing, planning and executing lessons. SCK (Specialised content knowledge) is the mathematical knowledge and skill unique to teaching. Teachers have to be able to do a kind of mathematical work that others do not. It involves unique mathematical understanding and reasoning and requires knowledge beyond that being taught to learners. Understanding different interpretations of mathematics problems and solutions and helping learners to make sense of their work and that of others is crucial. It involves making features of particular content visible to and learnable by learners and explaining how mathematical language is used. Teachers must choose, make and use mathematical representations that are effective and help students to explain and justify their mathematical ideas.

Horizon Knowledge, a provisional inclusion in the model, is an awareness of how mathematical topics are relayed over the span of mathematics included in the curriculum. This is about teachers understanding what the mathematics looks like in grades either above or below the grade they are instructing.

The category PCK, using this model, is comprised of knowledge of content and students (KCS), knowledge of content and teaching (KCT) and knowledge of content and curriculum. KCS (knowledge of content and learners) is knowledge that combines knowing about learners and knowing about mathematics. It involves anticipating what learners are likely to think and what they will find confusing as well as possible difficulties they may experience. It includes recognising errors and identifying most likely errors learners make, interpreting learner responses and developing learner justifications. Teachers have to know how to select lessons and assessment tasks that are interesting and motivating for learners. KCT (knowledge of content and teaching) combines knowing about teaching and knowing about mathematics. Teachers have to have a mathematical knowledge of the design of instruction. They must be able to sequence particular content for instruction and select and use effective instructional models. They have to identify what different methods and procedures to use to make the mathematics salient and usable by learners. This involves making decisions about what learner contributions to pursue and which to ignore or save for a later time. Knowledge of content and curriculum which is linked to Shulman's curricular knowledge is currently placed within pedagogical content knowledge and is supported by later work of Shulman and

colleagues. The categories of KCT and KCS also coincide with Shulman's two central dimensions of PCK.

Ball et al. (2005: 17) have rigorously tested this professional knowledge of mathematics through generating measures of teacher mathematical knowledge and linking these measures to learner performance. They have developed 250 multiple-choice items designed to equally measure teachers' common content knowledge (CCK) and specialised content knowledge (SCK) for teaching mathematics with the aim of identifying the knowledge needed for effective practice and to build measures of that knowledge. The measures cover the mathematical domains of number and operations, patterns, functions and algebra and most recently, some geometry items that were written by a range of experts – mathematics educators, mathematicians, professional developers, project staff and classroom teachers. Building a good item from start to finish: reviewed, revised, critiqued, polished, pilot tested and analysed, takes over a year. Their research thereby seeks to identify professional knowledge of mathematics that matters for the quality of teaching and attempts to produce tried and tested evidence that supports this claim.

Two sample assessment items, taken from the work of Ball et al. (2008: 399, 2005: 43), are included below to illustrate the distinction between CCK and SCK and the different focus of each assessment task.

Ms. Dominguez was working with a new textbook and she noticed that it gave more attention to the number 0 than her old book. She came across a page that asked students to determine if a few statements about 0 were true or false. Intrigued, she showed them to her sister who is also a teacher, and asked her what she thought. Which statement(s) should the sister select as being true?

(Mark YES, NO, or I'M NOT SURE for each tem below)

	Yes	No	I'm not sure
a) 0 is an even number.	1	2	3
b) 0 is not really a number. It is a placeholder in writing big numbers.	1	2	3
c) The number 8 can be written as 008.	1	2	3

Figure 2.3: : Item example for measuring common content knowledge (CCK)

Student A	Student B	Student C
$\begin{array}{r} 35 \\ \times 25 \\ \hline 125 \\ + 75 \\ \hline 875 \end{array}$	$\begin{array}{r} 35 \\ \times 25 \\ \hline 175 \\ + 700 \\ \hline 875 \end{array}$	$\begin{array}{r} 35 \\ \times 25 \\ \hline 25 \\ 150 \\ 100 \\ + 600 \\ \hline 875 \end{array}$

Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Which of these students is using a method that could be used to multiply any two whole numbers?

	<i>Method that would work for all whole numbers</i>	<i>Method that would NOT work for all whole numbers</i>	<i>I'm not sure</i>
a) <i>Student A</i>	1	2	3
b) <i>Student B</i>	1	2	3
c) <i>Student C</i>	1	2	3

Figure 2.4: Item example for measuring specialised content knowledge (SCK)

The work of Ball and colleagues (2004; 2005; 2008) is not an attempt to replace Shulman's work on content knowledge but rather to extend the theory through practice-based research. It develops in more detail "the fundamentals of subject matter knowledge for teaching" through research in the practice of teaching, by "elaborating sub-domains", and by measuring and validating knowledge of these domains (Ball et al., 2008: 402).

It is clear from their research that knowledge for teaching is multi-dimensional and although the MKfT model categories appear static, the teaching context, the mathematical content and experience of the teacher will blur the categories somewhat. At times it can be difficult to decide where one category starts and another finishes. However, what this mapping does do is to try to elaborate more specifically on what knowledge is required for teaching and to design suitable items to measure the different categories of knowledge. It recognises that definition and precision within each category is problematic; however, it is a useful framework to

begin to understand what mathematics teachers need to know for the task of teaching.

The emphasis on mathematical knowledge for teaching is not an attempt to eliminate the need for disciplinary mathematical knowledge but, as others have recognised, more high level mathematics may not necessarily produce better learners or improve teaching and learning of mathematics (Ma, 1999 and Adler, 2002). Teachers need to know the subject they teach from different perspectives. They must be able to do the mathematics in the curriculum, they must be able to plan, sequence and organise mathematical tasks and be able to recognise the conceptions and misconceptions of learners. It is crucial and most important that they know and are able to “use the mathematics required inside the work of teaching” (Ball et al, 2008: 403).

Petrou and Goulding (2011) highlight two principle concerns about the MKfT model. Firstly it does not acknowledge the importance of teachers’ beliefs about their teaching. Teacher beliefs about the nature of mathematics may be linked to their subject matter knowledge and how they approach mathematics teaching. Some teachers believe that mathematics is about procedures and rules and neglect to explore the conceptual understanding within the mathematics. This could be particularly relevant in the development of syntactic knowledge: conjecturing, finding evidence and seeking explanations. Petrou and Goulding (2011) argue that if the model is being used to describe the knowledge needed by teachers for teaching based on their actions then surely their belief about the nature of mathematical knowledge is relevant. Secondly, the domain of SCK is a central concept within the model of teacher mathematical knowledge but it does not clearly distinguish between SCK and PCK. SCK is defined as “the mathematical knowledge that is used in classroom settings and needed by teachers in order to teach effectively” while PCK is “a special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding” (Petrou & Goulding, 2011: 17).

While the work of Ball et al. has focused on the description of the mathematical work of teaching; other researchers within mathematics education are also exploring the ‘what’ of teacher knowledge in different ways. Baumert et al. (2010), a group of researchers in Germany, have begun to conceptualise and measure the

professional knowledge of secondary mathematics teachers. Their project, COACTIV (Professional Competence of Teachers, Cognitively Activating Instruction, and the Development of Students' Mathematical Literacy), involved a large-scale study of Grade 10 mathematics teachers and their learners to investigate the implications of content knowledge (CK) and pedagogical content knowledge (PCK) on the quality of instruction and learner progress. In contrast to the Ball et al. (2008) classification of mathematical knowledge for teaching as an amalgam of different types of mathematical knowledge, they distinguish between the CK and PCK of secondary mathematics teachers both conceptually and empirically. CK is conceptualised as "a profound mathematical understanding of the curricular content to be taught" and "has its foundations in the academic reference discipline" while PCK forms a "distinct body of knowledge that makes mathematics accessible to learners" (p. 142). PCK comprises three dimensions: knowledge of mathematical tasks, knowledge of learners' ideas and knowledge of representations and explanations. While their results confirm the relevance of CK and PCK for high quality teaching and learner progress, they found that PCK "largely determines the cognitive structure of mathematical learning opportunities" and CK had "lower predictive power for learner progress" (p. 166).

Adler and colleagues' research on mathematics for teaching in the QUANTUM project in South Africa, is also concerned with the mathematics (how much and what kind) teachers need to know and know how to use it to effectively teach in diverse classroom contexts (Adler & Davis, 2006).

2.2.2. Mathematics for Teaching (MfT)

Adler's (2002: 136) initial research in the context of a teacher in-service course addresses the issue of disciplinary knowledge for teaching and calls it *teachers' conceptual knowledge-in-practice*. While she acknowledges the crucial role of SMK in developing learners' high-level conceptual thinking she also recognises that *teachers' conceptual knowledge-in-practice* is another aspect of teacher knowledge. She defines *teachers' conceptual knowledge-in-practice* as a special kind of knowing about a subject that is substantially different from the kind of knowing and knowledge held by an expert in the discipline and later aligns with Ball et al.(2004) to conceptualise *mathematics for teaching* (MfT).

Adler & Davis's (2006) more recent work within the QUANTUM project highlights the serious tension and lack of engagement between research driven by pedagogical concerns and research driven by subject concerns, and the tension in foregrounding mathematics or pedagogy. One of the key goals of their project is the theoretical and methodological elaboration of mathematical knowledge for teaching. A critical element of knowing and doing mathematics in and for teaching is unpacking or decompression. Unpacking requires understanding the mathematics of a concept (epistemic), how this might develop in learning (cognitive) and the relationship between the two (Adler & Pillay, 2007). Ball and Rowan (2004) suggest knowing mathematics in detail means being able to unpack or decompress mathematical ideas so that they are accessible to learners and may be one of the essential and distinctive features of knowing mathematics for teaching (Kazima et al, 2008). However for Adler and Davis (2006) the underlying assumption in QUANTUM is that mathematics for teaching is situated in "pedagogic practice" and its constitution needs to be explored within a range of contexts "both in mathematics teacher education and in school mathematics teaching" (Adler & Pillay, 2007: 86).

In the research done by Adler and Pillay (2007), they found that through a broad analysis of a single teacher's teaching and discussion of his teaching, they could describe MfT as it comes to be constituted in practice. They acknowledge that general elements of MfT, as described by Shulman (1986) and Ball et al. (2008), can be identified and interpreted in a variety of different contexts but they take on a specificity in particular pedagogic practices. This is further elaborated in the work of Kazima et al. (2008: 284) with reference to mathematics for teaching (MfT) as "specialised mathematical knowledge that teachers need to know and know how to use in their teaching". They argue since the boundary between SCK and PCK is not clear in the practice of teaching, it is necessary to provide a more inclusive notion of mathematics for teaching. Their research with teachers in classrooms shows that MfT is a function of the "topic being taught", the "teachers' approach to teaching", and how "ideas are introduced" (p. 296). It confirms that the tasks identified by Ball and Rowan (2004) are not only "mathematical however they found that they take on specific meanings across topics and across different approaches to teaching" (p.296).

Davis and Renert (2009) offer a different perspective on mathematics for teaching with a focus on mathematics and the nature of the mathematical knowledge

embodied and *enacted* in teaching. They are interested in the images and metaphors that frame teachers' enacted understanding of mathematics and make a careful examination of the mathematical concepts teachers possess. Teachers use explicit procedures in teaching mathematics and are often not aware of the implicit associations that exist in their practice. This means that a teacher may cut off opportunities to develop learners' mathematical thinking through closing off alternative possibilities and over-emphasising the memorisation of rules and prescribed methods. They argue the need for teachers to have "knowledge of logical structures, figurative dimensions, and associations of mathematical concepts, combined with knowledge of mathematics teaching strategies" (Even, 2009: 147).

While the mathematics for teaching perspective highlights important issues relative to mathematics and the work of teaching, it is not being used in this study as it does not consider the link between SMK and PCK and its development within the context of preservice teacher education.

2.2.3. Mathematics Knowledge in Teaching (MKiT)

MKiT is the result of an in-depth examination of the development of thinking about mathematical knowledge and teaching described in the book 'Mathematical Knowledge in Teaching' (Rowland & Ruthven, 2011). Two aspects of the research discussed in this book namely the *Knowledge Quartet* (Rowland, Thwaites & Huckstep, 2003, 2006) and *Mathematical modes of enquiry* (Watson & Barton, 2011) are particularly relevant to this research and expand some of the issues addressed previously. Both examine teachers' mathematical content knowledge but from different perspectives. The first describes a practice-based framework of teacher knowledge-in-use in the course of actual teaching and is designed to provide a guide to the analysis and development of mathematical knowledge-in-use to support teachers' professional reflection and learning (Ruthven, 2011). The second outlines a contrasting view of MKiT as a way of being and acting within the mathematical practices of the classroom and explains how some of the tasks of teaching can be seen as particular applications of *mathematical modes of enquiry* (Watson, 2008).

a) The Knowledge Quartet (KQ)

The Knowledge Quartet (KQ) is an empirically-based conceptual framework for classifying ways in which preservice student teachers' SMK and PCK come into play

in the classroom. It can be used to identify, describe and analyse mathematical content knowledge 'enacted' in teaching and to provide a structure for reflection and discussions of lessons (Turner & Rowland, 2008).

The model arose from the findings of the SKIMA project, run by a group of researchers at the University of Cambridge, to investigate the mathematical content knowledge (SMK and PCK) of preservice primary teachers in England and Wales. They observed and videoed lessons of preservice teachers in a 1-year Post Graduate Certificate in Education (PGCE) course and used the framework to classify ways in which the teachers' SMK and PCK *played out* in the classroom (Rowland et al., 2009). They suggest that mathematical content knowledge is a complex combination of different types of knowledge that interact with each other and can be more easily seen in the action of teaching. Teachers require knowledge in several different domains but this research supports the belief that mathematical content knowledge needed for teaching is not only located in the mind but also through the practice of teaching.

The model is an elaboration of the earlier work by Shulman (1986) and a response to the work of Fennema and Franke (1992) by suggesting ways in which teachers' SMK relates to their PCK, and how their actions in the classroom are informed by their knowledge (Goulding & Petrou, 2008). The approach is different to the Ball model in that it does not try to see SMK and PCK as unique entities but rather as inter-related and dynamic constructs. It forms a link between theory and practice through the study of the act of teaching and can help preservice teachers identify aspects of their content knowledge that affects their teaching and how teacher knowledge could be improved (Rowland et al, 2009). The framework can also be used by teachers, tutors, mentors and teacher educators to give constructive feedback on how content knowledge affects teaching and offer suggestions on how to develop it.

The Knowledge Quartet framework is a tested descriptive and analytical tool and has been exposed to extensive theoretical sampling (Ruthven & Rowland, 2011). There are 18 descriptive codes which emerged from the analysis of 24 videos using a grounded theory approach. Researchers identified significant teacher action which was informed by SCK and/or PCK, then codes (as descriptors) were allocated and discussed until consensus was reached. The codes are a way of looking at and talking about mathematics teaching in practice with a focus on teachers and their

mathematical knowledge for teaching. The number of codes could make it difficult to observe and reflect on a lesson so they were grouped into a smaller set of “big idea” categories which was then called the *Knowledge Quartet* (Rowland et al., 2009: 28).

The Knowledge Quartet consists of four dimensions/quadrants, each with its own distinctive nature: Foundation, Transformation, Connection and Contingency (Rowland et al., 2009: 29-32; Turner and Rowland, 2011: 200-202).

Foundation

The first member of the quadrant is the teacher’s theoretical background and beliefs. This entails the knowledge possessed by the teacher and acquired at school, at college/university, including initial teacher education. It is the foundation of the other three dimensions of the quartet: transformation, connection and contingency. There are three key components to this dimension: knowledge and understanding of mathematics; knowledge of mathematics pedagogy acquired through systematic enquiry into the teaching and learning of mathematics; and beliefs about mathematics, including beliefs about why and how maths is learnt. The codes used to identify and describe how this knowledge ‘plays out’ in the classroom are: overt subject knowledge; theoretical underpinning of pedagogy; awareness of purpose; identifying errors; use of terminology; use of textbook and reliance on procedures. The next three dimensions refer to the ways and contexts in which knowledge is used in the preparation and act of teaching: they focus on knowledge-in-action as demonstrated both in planning to teach and in the act of teaching itself (Rowland et al., 2009: 30).

Transformation

This gets to the heart of what it means to teach a subject and looks at how the content knowledge of the teacher is transformed to enable someone else to learn it. It is what Shulman defines as the PCK base for teaching and is demonstrated in the planning of the lesson and in the act of teaching through the choices of representations, illustrations and explanations given and the questions asked from learners. The focus is on the learner(s) and the selection and use of examples and exercises to assist the development of mathematical concepts, to demonstrate procedures and to develop appropriate language.

Connection

The next category concerns the coherence and connections in the planning and/or teaching of the mathematics lesson or lessons. It includes making structural mathematical connections between and within concepts and procedures and making decisions about the sequencing of materials for instruction. The teacher needs to be aware of the cognitive demands of the task selected and its conceptual appropriateness.

Contingency

The final category deals with the teacher's response to unexpected and unplanned for events which occur within a lesson. They require the teacher to be able to take *contingent action* (think on one's feet) when something unanticipated happens. This involves responding to learners' questions and ideas, working with their responses, handling errors and misconceptions that were not part of the intended lesson. This is not easy for teachers and the level and depth of response can depend to some extent on the teacher's subject-matter knowledge, and is especially difficult for novice teachers.

The preceding description of the four dimensions of the Knowledge Quartet is summarised by Figure 2.5 (Rowland et al., 2009).

Dimension	Contributory codes
Foundation	awareness of purpose; adheres to textbook; concentration on procedures; identifying errors; overt display of subject knowledge; theoretical underpinning of pedagogy; use of mathematical terminology.
Transformation	choice and use of examples; choice and use of representation; use of instructional materials; teacher demonstration.
Connection	anticipation of complexity; decisions about sequencing; making connections between procedures; making connections between concepts; recognition of conceptual appropriateness.
Contingency	deviation from agenda; responding to students' ideas; use of opportunities; teacher insights during instruction

Figure 2.5: The Knowledge Quartet – dimensions and contributory codes

The Knowledge Quartet framework continues to be used for different purposes in a variety of contexts: for lesson observation and for mathematics learning development in England (Rowland, 2007); initial teacher education in Ireland (Corcoran & Pepperell, 2007); understanding relationships between mathematical knowledge and teaching in Cyprus (Petrou, 2009); as a tool to support reflections and discussions of preservice teachers about mathematics teaching and to develop participants' mathematical content knowledge for teaching (Turner & Rowland, 2011).

b) Mathematical modes of enquiry

There has been an interesting elaboration of the research on theoretical models of knowledge for teaching which extends the concept of *what* teachers need to know to include the *how*, that is “how they know it, how they are aware of it, how they use it, and how they exemplify it” (Watson & Barton, 2011: 67). This research suggests that while models of teacher content knowledge might be useful to extend understanding and descriptions of SMK and PCK, they risk ignoring what teachers do in relation to mathematics: *how teachers enact mathematics*. Teachers need to act as mathematicians themselves if they hope to encourage learners to *mathematise* and to create a mathematical environment where mathematics is performed. There needs to be opportunity for learners to develop different modes of enquiry: conjecturing, hypothesising, justifying, predicting, abstracting, generalising and proving as enacted in the classroom by the teacher. This is not about the teacher learning about modes of enquiry but demonstrating and using these in a supportive learning environment in the classroom. Barton's (2009: 7) contribution to the discussion of mathematical knowledge for teaching elaborates the aspect of “how a teacher must know” mathematics. He acknowledges that it is important to have knowledge in the sense of “knowing that” (angles on a straight line), “knowing about” (angles), “knowing how” (solve a problem), and “knowing to” (make conjectures). However there is a missing component of the mathematical knowledge of a teacher which is broader and more fundamental: how teachers hold their mathematics. “How teachers hold mathematics” is about a teacher's attitude and orientation towards mathematics and is comprised of four parts: a teacher's vision of mathematics, a teacher's philosophy of mathematics, a teacher's sense of the role of mathematics in society and a teacher's orientation towards the subject (VPRO). Barton suggests teachers need to develop their own VPRO for

mathematics and those of learners without being too rigid, and should be open to the alternate and developing views of their learners.

There is the suggestion that the notion of mathematical knowledge for teaching needs to be expanded to include various aspects of content knowledge, teacher as mathematician, teacher as creator of a mathematical environment and teacher who “holds” mathematics. This is located within the belief that without understanding more about how mathematical knowledge is brought to bear on the tasks of teaching, descriptions and audits of necessary knowledge are hypothetical/theoretical (Watson & Barton, 2011).

2.2.4. Conclusion for teacher knowledge

The results of research within teacher knowledge offered so far highlight different positions in thinking about mathematical knowledge for and in teaching. It does not attempt to address all aspects of mathematical subject knowledge but tries to give an overview of some of the predominant perspectives in the field.

The results of the empirical research discussed within the construct teacher knowledge are now organised and summarised into three categories taken from Ruthven (2011): *subject knowledge differentiated*; *subject knowledge interactivated* and *subject knowledge mathematised*.

Firstly *subject knowledge differentiated* is concerned with the identification of the types of subject related knowledge that are unique to teaching and can be used to form a taxonomy of such knowledge. The goal of such research is to present an “overarching heuristic framework that can guide the analysis, assessment and development of professional knowledge” (p. 83). It started with the work of Shulman and has been further developed in different contexts described in some of the earlier models.

The work of Ball and colleagues falls within this category and is useful in helping understand what teachers need to know and how to use it in the teaching of mathematics. They provide a description and measure of the knowledge entailed in teaching and transform the work of Shulman into a more operational form. Their tasks of teaching (defining, explanation, working with learner ideas, representations, restructuring tasks and questioning) are useful in drawing focus to the work of

teachers (tasks of teaching) in the classroom and help formulate descriptions of what this means within the South African context. The tasks will be used in this study as a guide to track the development of preservice teachers' knowledge for teaching early algebra.

Rowland et al. (2009) provide a useful framework for the analysis of preservice student teachers' knowledge for teaching by focusing on four aspects of knowledge (SMK and PCK) within the lesson: foundation, transformation, connection and contingency. Rowland et al. (2009) alert us to the fact that though the Ball items are helpful and can give a measure of theoretical PCK, they might not necessarily reflect how a teacher acts in practice. The Knowledge Quartet provides a heuristic that helps guide attention to, and analysis of, mathematical knowledge-in-use within teaching (Ruthven, 2011).

Subject knowledge interactivated focuses on the evolution of thinking about the character of mathematical knowledge and how it is mediated through teaching and learning and is concerned with the processes through which mathematical knowledge is (re)contextualised and (re)constructed in the classroom.

Mathematics in teaching (MiT) reminds us that mathematics for teaching is situated in 'pedagogic practice' with no clear boundary between SMK and PCK. While it is acknowledged and relevant to the work of teaching, it is not the focus of this particular study, as this research is not looking at the character of mathematical knowledge and its mediation through teaching.

Subject knowledge mathematised calls for a different kind of knowledge than the type emphasised in mathematical knowledge for teaching hitherto. It is concerned with the range of "intellectual dispositions and strategies which can be thought of as *mathematical modes of enquiry*" (Ruthven, 2011: 91). They are also known as mathematical habits of mind in different areas of mathematics and refer to the mathematical processes teachers and learners need to operate within mathematics, teaching and learning. Barton (2009: 7) suggests that transforming different "aspects of mathematical knowledge into effective teaching lies in a teacher's attitudes and orientations towards mathematics: the way they hold their mathematics, the way they know mathematics and their relationship with mathematics".

The empirical research discussed within the construct of teacher knowledge gives an overview of the different aspects and the commonalities and differences within the different categories. The next section of this literature review deals with the second construct of this research, namely early algebra.

2.3. Early Algebra

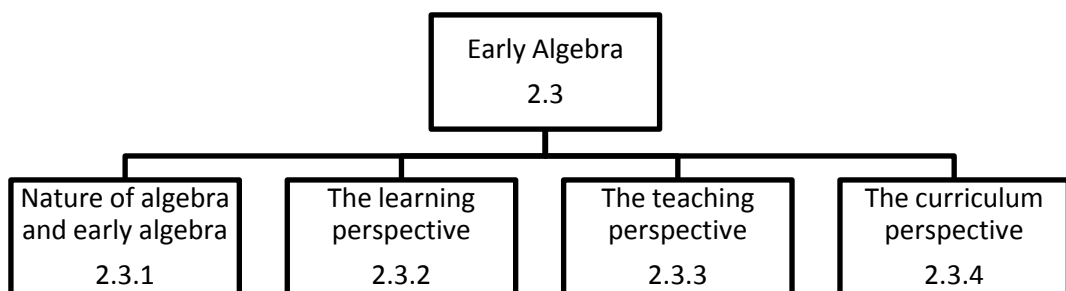
The mathematics children learn in primary school lays the foundation for high school mathematics. Algebra forms a large component of the mathematics curriculum for high school and for further study in many mathematics related courses. It seems logical to ensure that young children are building and developing the concepts and skills necessary for future success in algebra.

Examination of mathematics curricula across various countries (Mullis et al., 2003) and the structure of mathematics appears to suggest that mathematics needs to progress on a continuum from simple to more complex computations. Indeed the majority of mathematics textbooks support this approach and focus on fluency with whole number and particular aspects of geometry and measurement, as precursors for algebra learning (NMAP, 2008). This position favours the traditional and often prevailing notion of an algebra learning trajectory of arithmetic followed by algebra, seen as “distinct subject matter standing in a particular order” (Schliemann, Carraher & Brizuela, 2007: x). Arithmetic and algebra are traditionally treated as separate topics with some bridging in-between called pre-algebra. Hercovics & Limboski (1999) identified a cognitive gap which appears to emerge in the pre-algebra stage which is the result of shifting from arithmetic thinking to algebraic thinking without recognising the algebraic structure within arithmetic.

Research on the difficulties of moving from arithmetic to an algebraic form of reasoning has informed current work on the emergence of algebraic thinking in the earlier grades (Kieran, 2004). Current research challenges the traditional approach to teaching algebra and reports on a growing alternative perspective which integrates arithmetic and algebraic thinking from the early grades. The wider consideration of the work of other mathematics educators and researchers offers alternative ways of conceptualising school algebra which will be elaborated further herewith.

A substantial body of research has been written about algebra and early algebra over the past 20 years: it has been the focus of numerous conferences, journal articles, chapters and books. There has been a number of research projects which have looked at various aspects of early algebra including the STARR project (Supporting the Transition from Arithmetic to Algebraic Reasoning in elementary and middle school) (Borko & Clarke, 2005a); the Early Algebra Project at TERC and Tufts universities (Carraher et al, 2006, 2008a, 2008b); GEARR (Generalising to Extend Arithmetic to Algebraic reasoning (Blanton & Kaput, 2003, 2004, 2005a, 2005b) and the Early Algebra project in Australia (Warren & Cooper, 2001). There have also been special journal editions focussing specifically on early algebra: The Mathematics Educator Singapore (2004); ZDM (2005); and Mathematics Thinking and Learning Journal (2007). There have also been international reports: The Future of the Teaching and Learning of Algebra – the 12th ICMI study (Stacey & Chick, 2004); the MAA report: Algebra - Gateway to a Technological Future (Blanton et al, 2007) and NMAP: Foundations for Success report (2008).

The projects, reports, journal articles and books considered here cover a broad range of studies in early algebra and the development of algebraic thinking in earlier grades from a learning and instructional perspective as well as curricular coverage across a variety of different countries: US, Singapore, China, Hong Kong, England and South Africa. (Cai & Knuth, 2005; Kaput & Blanton, 2008).



The intention of this part of the literature review is to highlight examples of recent studies in the field of early algebra and to establish a meaningful understanding of early algebra and its related components: learners, teacher and the curriculum. Section 2.3.1 starts with a rationale for the study of early algebra and outlines the distinctions between school algebra (traditional high school algebra) and early algebra (algebraic thinking in the primary school). This is followed with an explication of the different meanings of early algebra taken from diverse

perspectives and concludes with a definition of what early algebra means in the context of this study. The sections 2.3.2 – 2.3.4 link to the design principles of the Maths 2 course for the preservice teachers. A goal of the Maths 2 was to develop connections between different knowledge domains (Thanheiser et al, 2010). The instructional strategies used to achieve this goal were to connect knowledge of content and teaching through explication of the lecturer actions, to include artefacts of learners' mathematical thinking which helped connect knowledge of content and students, and to connect knowledge of content and the curriculum through the design and teaching of early algebra lessons. This section (2.3) contains an overview of the field of empirical research in the early algebra from a learning, teaching and curricular perspective across different countries which highlights important issues within early algebra teaching and learning.

2.3.1. Nature of algebra and early algebra

As mentioned previously there are numerous studies highlighting the difficulties learners have in learning algebra in the high school. There is a growing body of research which suggests that the development of algebraic thinking in the earlier grades could help alleviate this problem (Carpenter et al., 2005; Carraher et al., 2008; Kaput & Blanton, 2008; Warren, Mollinson & Oestrich 2009). If learners and teachers regularly spend the first six years of primary school simultaneously developing arithmetic and algebraic thinking, the study of algebra later would become a “natural and non-threatening extension of the mathematics of primary school” (Cai & Moyer, 2008: 3). This is an alternative approach to the traditional approach to the teaching of algebra, and is based on the work of mathematics educators and researchers. This could promote a stronger integration of algebraic thinking and reasoning in the foundational grades. Kaput (1998) believes it is necessary to reconceptualise the nature of algebra and algebraic thinking and to re-examine when children are capable of algebraic thinking and when these ideas need to be introduced into the curriculum. He believes separating arithmetic and algebra deprives children of the chance to develop powerful schemas for thinking about mathematics and makes the learning of algebra more difficult in the later grades. The focus on developing algebraic thinking can prepare learners for “increasing complex mathematics and cultivates habits of mind that attend to the deeper underlying structure of mathematics” (Blanton & Kaput, 2005a: 412).

There is evidence to suggest that learners with limited experience of arithmetic operations cannot master algebra sufficiently and an immature understanding of arithmetic operations hinders their ability to learn algebra. Lins and Kaput (2004) argue for an early start to algebra instruction which provides a special opportunity to develop a particular kind of generality in learners' thinking. This does not mean the same old school algebra but rather a new algebraic way of thinking and reasoning, an immersion in the culture of algebra. Algebra requires a firm understanding of operations. Reconceptualising arithmetic to stress its algebraic nature requires a structural (relational) understanding of mathematical statements, and this can be developed in the context of arithmetic to better prepare learners for algebra (Stephens & Fujii, 2002).

Early algebra is not an attempt to introduce symbol manipulation earlier to younger children, but rather an attempt to reform and update the teaching of arithmetic in a way that stresses its algebraic character. It requires understanding how the arithmetic concepts and skills can be better aligned with the concepts and skills needed in algebra so that learning and instruction is more consistent with the kinds of knowledge needed in the learning of formal algebra (Carpenter et al, 2005). Learners in the early grades can begin to engage in meaningful discussion about mathematical proof and make significant progress in understanding its nature and importance. The development of their algebraic reasoning will be reflected in their ability to generate, represent, and justify generalisations about fundamental properties of arithmetic. (Carpenter, 2003)

The early development of algebraic reasoning also requires a better understanding of the various factors that make the transition from arithmetic thinking to algebra difficult for learners. Bridging the gap from arithmetic thinking to algebraic thinking is problematic for children (Booth, 1988; Herscovics & Linchevski, 1994; Bednarz, Kieran & Lee, 1996). The concepts and skills needed to think algebraically have often not been developed and the traditional approach to teaching algebra assumes too many foundations. Linking arithmetic and algebraic thinking in the early grades could help develop the necessary skills for future success. The research findings of Malara (2004) suggests the mental models of algebraic thinking and language should be constructed in an arithmetical environment, even from the very first years of primary school, teaching children to think of arithmetic in an algebraic way.

a) Formal algebra and links to Early Algebra

Most high school graduates have a traditional understanding of formal algebra as symbol manipulation and wonder why this knowledge is important. Simultaneously there is a general belief that algebra education is something common around the world when in fact the differences are now known to be large on all dimensions – the degree of formalism, the amount of manipulation, the place of functional thinking, the use of technology, the age of introduction, etc. (Kendal & Stacey, 2004). Stacey (2006) identifies two major external social and political considerations: massification of education and new technology which has led mathematics educators to rethink and research the nature of algebraic activity, the instructional approaches to the subject and learner needs.

A cursory review of the mathematics education literature characterises algebra in different ways. For example, Usiskin (1988: 18) describes four conceptions of algebra:

1. Generalised arithmetic e.g. $a + b = b + a$ (variables as pattern generalisers: translate, generalise)
2. The study of procedures for solving certain kinds of problems e.g. $5x + 3 = 40$ (variable as unknown or constants: solve, simplify)
3. The study of relationships among quantities e.g. $A = l \times b$ (variables vary: relate, graph) [function ideas]
4. The study of structures e.g. factorise $3x^2 + 4ax - 132a^2$ (variable is little more than an arbitrary symbol: manipulate, justify)

Bednarz et al. (1996) suggest four conceptual approaches to developing learners' algebraic ideas: generalisation, problem solving, modeling and functions:

1. Generalisation focuses on the construction of a formula that accounts for general procedures or relationships among quantities e.g. what would be the area and perimeter of the 11th triangle in a given sequence and what is the "nth" term?.
2. Problem solving emphasises the forming and solving of equations by using letters as unknowns e.g. $x + z = 8.5$, find x .
3. Modeling involves algebraic representations arising out of real world situations, and relationships originating in observations or measurements.

4. Functional approach examines various representations of dependent relationships among real-world quantities, observing how change in one variable produces variation in another.

There is a further categorisation by Kieran (2004: 141-142) which proposes a 3-part model of algebraic activity, according to the typical activities engaged in by students. It consists of the following types of activity:

1. Representational activities

Much of the meaning-building for algebraic objects occurs within these activities and involves:

- i) equations containing an unknown that represents problem situations, e.g. there are 3 piles of stones; the first has 5 less than the third, and the second has 15 more than the third. There are 31 altogether. Find the number in each pile;
- ii) the expressions of generality arising from geometric or numeric sequences, e.g. looking at the numerical and geometrical pattern and stating a rule for extending the sequence of pictures indefinitely; and
- iii) expressions of the rules governing numerical relationships e.g. the sum of two consecutive numbers is always an odd number. Can you show why, using algebra?

2. Transformational activities

This is concerned with changing the form of an expression or equation to maintain equivalence e.g. collecting like terms, factoring, expanding, substituting, solve and simplify.

3. Global meta-level activities

These activities involve algebra being used as a tool but are not exclusive to algebra and could be engaged in without any algebra at all e.g. problem solving, modelling, noticing structure, studying change, generalising, analysing relationships, justifying, proving and predicting.

The current view of algebra is to see representational and transformational activity as interlinked which involve concepts and skills together and make an important contribution to the understanding of algebra (Brown & Drouhard, 2004).

While each of these characterisations of algebra highlights important aspects, Kaput (2008, 10-14) provides a useful framework, based on many years of work in the field of early algebra, for considering the various elements of algebra. He identifies two core aspects of algebra related to the reasoning processes and three strands of early algebra:

Core aspect A - Algebra as **systematically symbolising generalisations** of regularities and constraints i.e. using symbols to generalise.

Core aspect B - Algebra as **syntactically guided reasoning and actions** on generalisations expressed in conventional symbol systems i.e. acting on symbols, following rules.

Kaput's (2008) core aspects run through each of three strands of early algebra:

Strand 1: **Generalised arithmetic** - Algebra as the study of structures & systems in arithmetic and quantitative reasoning.

Strand 2: **Functional thinking** - Algebra as the study of functions, relations & joint variations.

Strand 3: **Modeling** - Algebra as the application of a cluster of modelling languages both in and out of maths.

Roberts (2010: 168) summarises the work of Kaput into a useful table provided in figure 2.6 overleaf:

	CORE ASPECT A: Using symbols to generalise	CORE ASPECT B: Acting on symbols and following rules
STRAND 1: Generalising from arithmetic and quantitative reasoning	<p>Generalising arithmetic: properties of zero, commutativity, inverse relationships</p> <p>Generalising about particular number properties and relationships; equivalence of the equal sign</p>	<p>Initial symbolisation that avoids numerals and uses symbols to then compare quantities: Comparison of lengths, or heights or volume.</p> <p>Leads later to abstract algebra</p> <p>Leads later to calculus and analysis</p>

	CORE ASPECT A: Using symbols to generalise	CORE ASPECT B: Acting on symbols and following rules
STRAND 2: Generalising towards the idea of a function	Towards functions: Recognising regularity through elementary patterning, ideas of change, linearity, change, representation through tables, graphs and 'function machines'	Multiple representation types
STRAND 3: Modeling as a language	Number-specific modeling: Arithmetic problem where variable is unknown. Situation is modelled, more variable express a class of functions Generalisation modeling: variable as parameters to explore effects in pure arithmetic word problems, algebrafying, multiple representations	

Figure 2.6: Kaput's definition of early algebra

The first two strands of this model consider two types of generalising which are at the heart of algebraic thinking: generalising arithmetic; and generalising towards the idea of function. The third strand refers to modeling processes where situations are understood and interpreted using algebraic reasoning and language. Although the strands are presented as unique entities there are overlaps between the strands and the table provides illustrative examples of each strand taken from Kaput (2008).

While formal algebra tends to focus on the algebra needed for high school, early algebra is concerned with the algebraic thinking that needs to be developed in the early grade to promote future success in algebra.

b) Early algebra – Definition

There is a general agreement that algebraic thinking in earlier grades extends beyond arithmetic and computational fluency to attend to the deeper underlying structure of mathematics (Cai & Knuth, 2005). Carpenter and Levi (2000) argue that making generalisations and using symbols to represent mathematical ideas and to represent and solve problems, are two central aspects of algebraic thinking.

Kieran (2004: 149) offers a definition of algebraic thinking in the early grades which is based on global-meta level activity of algebra as described in her 3-part model of the main activities of formal algebra (discussed under 2.3.1 a), to include new directions on current research on algebra in the early grades. It recognises the

functional approach which has begun to permeate algebraic activity and highlights its inclusion and emphasis in curricula of many countries. She proposes a framework for thinking about algebraic thinking in the early grades which is “grounded in actuality” and makes links to “existing frameworks for thinking about algebra in the later grades”:

Algebraic thinking in the early grades involves the *development of ways of thinking within activities* for which letter-symbolic algebra can be used as a tool but which are not exclusive to algebra and which could be engaged in without using any letter-symbolic algebra at all, *such as, analysing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modelling, justifying, proving, and predicting* (p. 149).

There is another focus, outlined by Carraher et al. (2006, 2008a, 2008b), on early algebra as generalised arithmetic of numbers and quantities which is a move away from computations on particular numbers. It involves seeing arithmetic operations as functions through thinking about relations among sets of numbers. They recognise arithmetic as an actual part of algebra and propose that primary school instruction must focus on bringing out the algebraic character of arithmetic.

This approach is illustrated by Figure 2.7 (Schliemann et al., 2007):

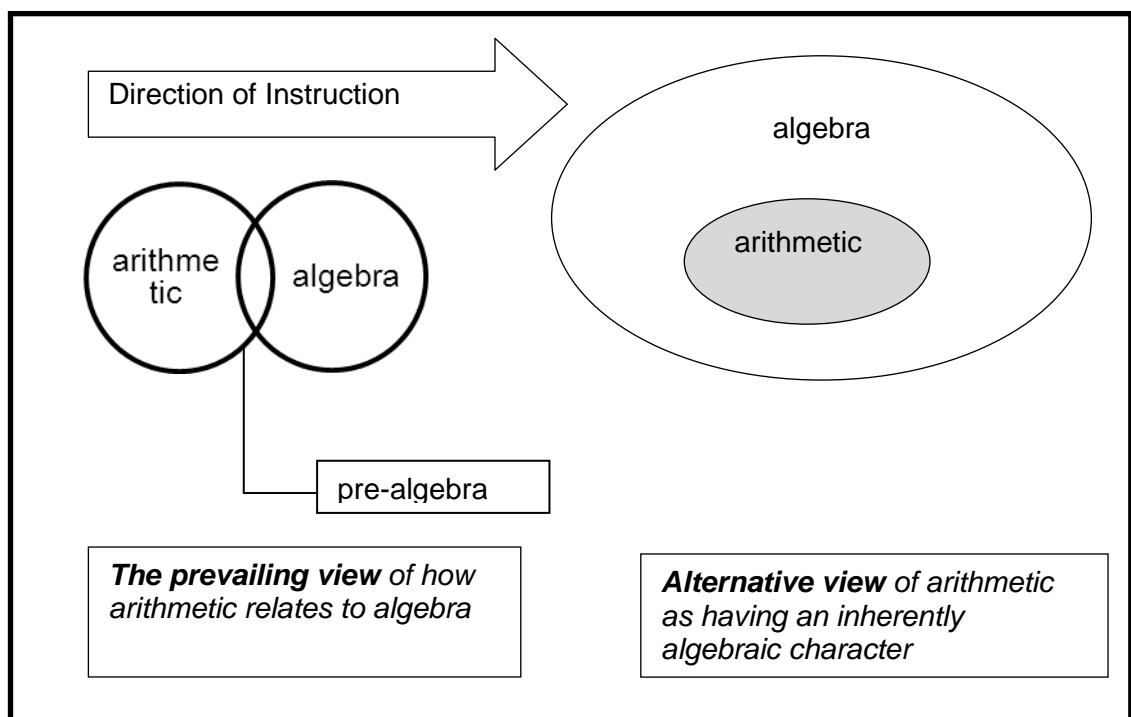


Figure 2.7: Rethinking algebra teaching

Warren et al. (2009: 10) and colleagues in Australia propose a similar view of early algebraic thinking and call for a reconsideration of how we think about arithmetic. There is a need to assist learners to engage with arithmetic thinking with a focus on structure rather than on arithmetic as a computational tool only. Multiplication tables can be seen as an operation involving computation, but limiting arithmetic thinking to computation only does not help children to have conversations that can lead to generalised arithmetic and eventually towards formal algebra.

Schoenfeld (2008: 482; 506) highlights the emergence of early algebra research in providing an opportunity to develop children's mathematical thinking and reasoning from early in their school career and cultivate habits of mind for understanding mathematics that will prevail through to the higher grades. He believes the fundamental purpose of EA should be to provide learners with a set of experiences that enables them to see mathematics – sometimes called the science of patterns - as something they can make sense of, and to provide them with the habits of mind that will support the use of the specific mathematical tools they will encounter when they study algebra. EA provides learners with the kinds of sense-making experiences that will enable them to engage appropriately in algebraic thinking. Algebraic thinking consists of the ability to make effective and purposeful use of symbols in ways that are inherently sensible and meaningful. It means having access to various forms of representations, and includes being able to operate on symbols meaningfully in context when needed.

The research discussed so far generally takes an approach to early algebra which encourages learners to reason from particular quantities towards building mathematical generalisations. However there is also an alternative approach to algebraic thinking which emerges from the Davydovian tradition in Russia outlined in Dougherty (2003). It uses the exploration of mathematical generality itself (rather than the particulars of number) as a springboard for building learners' understanding of mathematical structures. In this approach to early algebra, learners begin by comparing abstract quantities of physical measures (e.g. length, area, volume), devoid of quantification, in order to develop general relationships about these measures (e.g., the transitive property of equality).

While there is acknowledgement of the eclectic and diverse views of early algebra in the field of mathematics education, the work of pioneer researcher Kaput (2008) has

helped define the field in more specific terms. Blanton & Kaput (2005b) offer a definition of early algebra as a process in which learners generalise mathematical ideas from a set of particular instances, establish those generalisations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways. They later use the term early algebra to refer to algebraic thinking in the elementary grades which is designed to help children “see and describe mathematical structures and relationships for which they can construct meaning” (Blanton, 2008: 6).

Early algebra is about making important ideas of mathematics, particularly algebra, accessible and relevant to children and teachers (Blanton, 2008). Early algebra is not to be seen as additional work to the current curriculum requirements. It is not a topic to be taught after children acquire arithmetic skills and procedures but is developed parallel to the development of arithmetic knowledge. It is about developing a way of thinking and reasoning that benefits all aspects of mathematics and is not about introducing algebraic skills and procedures earlier in the primary school. Early algebra provides depth, meaning and coherence to children’s mathematical understanding and provides the opportunity to generalise relationships and properties in mathematics.

Early algebra can be approached as a generalisation from arithmetic and quantitative reasoning and alternatively as a generalisation towards the idea of a function (Kaput, 2008). This study uses the term early algebra to refer to algebraic thinking as it relates to functional thinking and works with preservice teachers and learners to develop algebra as the study of function, relations and joint variations. The Maths 2 course focused on developing the concept of function and established links to early algebra and while some conceptions of EA as generalised arithmetic were addressed, there was insufficient time for a proper exploration of the topic.

Building a practice to develop children’s’ algebraic thinking requires a significant process of change for primary school teachers, who are often schooled in traditional arithmetic ways of doing mathematics. Teachers need to develop *algebra eyes and ears* as a new way of both looking at the mathematics they are teaching and listening to learners’ thinking about it (Blanton & Kaput, 2003). Functional thinking offers a rich and accessible entry point for teachers to understand the development of algebraic thinking and to design suitable teaching and learning experiences. The

route to functional thinking generally starts with the investigation of numeric and geometric patterns.

- **Patterns**

The importance of patterns and functions as a route to formal algebra and an expression of generality have been widely recognised and acknowledged by researchers in the field of early algebra and provide an opportunity for genuine and substantial mathematical activity (Mason, 1996; Orton & Orton, 1999; Stalo et al., 2006; Kaput & Blanton, 2008; Schoenfeld, 2008). For many, pattern is the heart and soul of mathematics and it is crucial in a study of mathematics to focus learners' attention to the patterns that underlie many aspects of mathematics (Zazkis & Liljedahl, 2002).

Pattern activities have the potential to be a powerful tool for developing “understanding of the dependent relations among quantities that underlie mathematical functions” as well as a “concrete and transparent way for young students to begin to grapple with the notions of abstraction and generalizations” (Moss & Beatty, 2006: 193). Pattern tasks which lead to generalisation are important in making the transition from arithmetic to algebraic thinking and provide a useful introduction to the concept of variable and future work with symbols (Zazkis & Liljedahl, 2002; Stalo et al., 2006). Anthony and Hunter (2008) suggest using tasks involving numeric patterning and function activities which can offer an opportunity to integrate early algebraic thinking into the existing mathematics curriculum. Teaching algebra concepts in the early grades requires the use of patterns for making connections in three ways: using symbols to express generalisations; developing meaning of or reasons for those symbolic generalisations, and using other algebraic representations such as graphs and tables (Richardson, Berenson & Staley, 2009).

Mulligan, Mitchelmore & Prescott, (2006: 290) broadly define pattern as a “numerical or spatial regularity” in which “the relationship between the various components of a pattern constitute its structure”. They regard pattern and structure “as inherent or constructed from, brought to or imposed on mathematical systems.” In other words, patterns and structure are inherently linked to mathematical activity through the recognition of regularity and the description of the relationship through generalisation. The results of their classroom research indicate that young children

can be taught to “seek and recognise mathematical patterns and structures’ which appear to have a substantial effect on their overall mathematics achievement” (p. 214).

The South African curriculum for Foundation Phase mathematics (DBE, 2011a:10 - 11) includes patterns, functions and algebra as a key mathematics content focus:

A central part of this content area is for the learner to achieve efficient manipulative skills in the use of algebra. It also focuses on the description of pattern and relationships through the use of symbolic expressions, graphs and tables; and identification and analysis of the regularities and change in patterns, and relationships that enable learners to make predictions and solve problems.

The Foundation Phase (Grades 1-3) curriculum mentions that learners need to work with both number and geometric patterns and should use physical objects, drawings and symbolic forms to copy, extend, describe and create patterns focussing on the logic of patterns and lays the basis for developing algebraic thinking skills (DBE, 2011a: 10).

The Intermediate (Grade 4-6) extends numeric and geometric patterns with a special focus on the relationships between terms in a sequence and between the number of terms (its place in the sequence) and the terms itself. It includes developing the concepts of variables, relationships and functions to describe the rules generating the patterns. It has a particular focus on the use of different, yet equivalent, representations to describe problems or relationships by means of flow diagrams, tables, number sentences or verbally (DBE, 2011b: 6).

- **Pattern as Function (Functional Thinking)**

Functional thinking, as a strand, can help teachers to build from patterns to generality in their curriculum and instruction. Blanton and Kaput (2005b: 35) conceptualise functional thinking as “building and generalizing patterns and relationships using diverse linguistic and representational tools” and “treating generalised relationships, or functions, that result as mathematical objects useful in their own right”. They use three modes for analysing patterns and relationships, outlined by Smith (2008), as a framework to discuss the kinds of functional thinking which can be found in classrooms:

1. recursive patterning which involves finding variation within a sequence of values
2. co-variational thinking which is based on analysing how two quantities vary simultaneously and keeping that change as an explicit, dynamic part of a function's description (e.g., as x increases by one, y increases by three)
3. correspondence relationship which is based on identifying a correlation between variables (e.g. y is 3 times x plus 2).

Smith (2008: 143) describes functional thinking as “representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalisations for that relationship across instances”. He proposes a framework for functional thinking (representational thinking) which involves six activities that underlie functional thinking and the construction of functions:

Engaging in a problematic within a functional situation	<ol style="list-style-type: none"> 1. Engaging in some type of physical or conceptual activity. 2. Identifying two or more quantities that vary in the course of this activity and focusing one's attention on the relationship between the two variables.
Creating a record	<ol style="list-style-type: none"> 3. Making a record of the corresponding values of these quantities, typically tabular, graphical or iconic.
Seeking patterns and mathematical certainty	<ol style="list-style-type: none"> 4. Identifying patterns in these records 5. Coordinating the identified patterns with the actions involved in carrying out the activity. 6. Using this coordination to create a representation of the identified pattern in the relationship.

Figure 2.8: Framework for functional thinking

Smith (2008: 145) argues that functional thinking occurs when an individual engages in an activity and chooses to pay attention to two or more varying quantities, and it is the act of focusing on the relationship that is central to the concept of function. While the individual is crucial to the process, the teacher provides the opportunity for the individual to engage in functional thinking by “creating activities, describing variable quantities, and posing appropriate questions and engaging in classroom discourse” (p.145).

For Schliemann et al. (2008) functions, rather than equations, may be a more useful entry point for introducing algebra, especially among young learners. They predict

the concept of function can help unite seemingly disconnected topics in the early curriculum and give the example of arithmetic operations re-conceptualised as instances of functions; multiplication tables can be viewed as embodiments of the function $a \times b$ and a function such as $7x + 3$ becomes an extension of the "times 7" table. Functions can be used as a "gateway into algebra, the expressions on each side of the equation can be treated as functions; the equals sign constrains each function to values in the co-domain that map to the same value in the range" (p.1). Through a functions-based approach to teaching, learners learn to shift their attention from particular, 'actual' cases (e.g. the view that x has a determined value) to the set of possible cases.

While the use of patterns as a route to functional thinking may seem like a logical introduction to formal algebra, it is not without difficulty. Research findings indicate that learners extend patterns numerically more easily than they can generalise about them (Zaskis & Liljedahl, 2002, Carraher et al., 2008a, 2008b). Some learners tend to focus exclusively on the numeric aspect of the pattern activities even when the patterns are presented visually (Noss et al. 1997). Samson & Schafer (2007) also found that learners respond differently to pictorial pattern contexts and while some find it helpful, it can create additional complications for others.

Orton and Orton (1999: 120) conducted a major study on patterning with over 1000 learners, and identified three major obstacles to successful generalisation starting with arithmetical incompetence which inevitably obstructs progress. Secondly, learners tended to focus on difference between consecutive terms (recursion) which provided an obstacle to generalisation and prevented them from seeing the overall mathematical structure. Thirdly, many of the learners were inclined to rely on inappropriate solution strategies and often made use of idiosyncratic methods in unpredictable ways.

There is empirical evidence to support the notion that young children are capable of thinking functionally and suggests a variety of teaching strategies such as "using concrete materials to create patterns, specific questioning to make explicit the relationship between pattern and its position, and specific questioning that assist children to reach generalisation with regard to the unknown positions" (Blanton & Kaput, 2004; Warren, 2005: 311). However, there is also evidence which suggests

that learners tend to focus on pattern spotting in one data set rather than on the relationship between the pattern and its position. This may be the case either because “single variational thinking is cognitively easier for children or is so engrained from school experience that there is a tendency to revert to it” (Warren, 2005). These research findings would also suggest that some of the difficulties “are not so much developmental but experiential” hence the need to bridge these gaps in the early years (p. 312). Limiting pattern finding to single variable data has less predictive capacity and is not as mathematically powerful as functional thinking. There is a fundamental shift in thinking needed by early grade teachers to move from analysing single variable data (simple patterns) to two or more quantities simultaneously, in order to develop children’s functional thinking and subsequently grow their algebraic knowledge (Blanton & Kaput, 2004).

Mason (2008) promotes a more flexible view of patterns which helps learners to recognise patterns that lead to algebraic generalisation and symbolisation. He characterises algebra as a “succinct and manipulable language in which to express generality and constraints on that generality” (Mason, 2008: 77). He suggests that *expressing generality* is not a strategy to be taught, used and tested but rather a holistic approach to mathematics. It requires the construction of a curriculum that exploits and makes use of children’s natural power to make sense of their experiences, “especially to make sense mathematically” (p. 89). This means that learners need to be encouraged and trusted to develop their own thinking and understanding through pattern activities and experiences which are structured to encourage learners to make generalisations, to explain and justify their generalisations and to convince one another. He supports the notion of *scaffolding and fading*, as a way to improve pedagogical effectiveness and present learners with a varied diet of rich mathematical experiences.

This section (2.3.1) presented a rationale for early algebra, established a link between algebra and early algebra and concluded with a justification for a focus on early algebra as functional thinking. The next three sections (2.3.2; 2.3.3; 2.3.4) will look at results of empirical data from the learning, teaching and curricular perspective and how they inform and relate to this study.

2.3.2. The learning perspective

Research related to early algebra and learners from Grades 1 to 9 is prolific and has contributed greatly to this area of study. It has helped provide evidence of the capability of children to handle algebraic concepts in the early grades and helped define algebraic thinking.

This section considers findings from research related to algebraic thinking among learners from Grades 1 to 9. The results have been organised into three categories: development of learner algebraic thinking; learner capacity for algebraic thinking; and difficulties and misconceptions of algebraic concepts.

a) Development of learner algebraic thinking

A number of studies have looked at the development of learners' algebraic generalisations, justification strategies, and relational and functional thinking (Carpenter et al., 2005; Warren, Benson; Green, 2007; Anthony & Hunter, 2008; Hunter & Anthony, 2008). Results suggest generalisation strategies and relational thinking skills can be expanded through purposely designed tasks and teacher action. Some studies focus on developing and supporting algebraic thinking through patterning, exploration of the commutative principle, instructional design and relevant instructional contexts (Carpenter & Levi, 2000; Stacey, 2003; Blanton & Kaput, 2003; Warren, 2005; Mc Nab, 2006; Hunter, 2010). The findings suggest it is no longer appropriate to have an arithmetic curriculum which focuses exclusively on computation; there needs to be opportunity to include experiences of generalisation, mathematical structure and properties of operations that underpin algebra through purposeful teacher action (MacGregor & Stacey, 1999; Banerjee & Naik, 2008). One of the important ideas of early algebra is therefore to build intuitive awareness of general properties of number operations throughout schooling. Consider, for example, the algebraic relationship $a - (b - c) = a - b + c$. Children often learn this as a rule for changing sign, but it is also a property which quite young learners can use intuitively. The intention is not to introduce learners to the formal expression, but rather to have them experience the algebraic relationship as a type of number sentence that is true for all numbers and that such relationships can help in working with numbers. For learners to come to learn algebraic ways of thinking, they need a careful introduction which helps them to identify differences between the reasoning used for solving problems with algebra and the reasoning for solving problems with arithmetic.

b) Learner capacity for algebraic thinking

The Algebra in Early Mathematics project is adding to research in the area of early algebra and mathematics education and provides compelling evidence that young learners (9 - 11 years of age) can learn algebra as an integral part of early mathematics (Carraher et al., 2008a, 2008b). Their approach, developed over the past 10 years, focuses on algebra as a “generalised arithmetic of number and quantities” (Schliemann et al., 2003: 128). In other words, they do not see arithmetic and algebra as distinct, and propose a deep and thorough understanding of arithmetic that requires “mathematical generalisations and understanding of basic algebraic principles” (Carraher & Schliemann, 2007: 689). The introduction of algebraic activities in primary school is seen as a move from computations on numbers and measure towards thinking about relationships between numbers’ i.e. operations of arithmetic should be seen as functions. Many representations are included in this approach: natural language, line segments, function tables, Cartesian graphs and algebraic notation.

The research work of Carpenter et al. (2003) is also testament to the lack of connectivity between arithmetic and algebra. They provide empirical research which investigates children’s thinking and representations and shows they can successfully learn about rules, principles, and representations of algebra in the early grades. Cases in which learners have been given problems to solve which involve making predictions, guessing, justifying, conjecturing and proving have achieved success (Blanton & Kaput, 2003). Results show that even young children are able to generalise, provide examples of relations and functions, and describe and give reasons for inverse relationships (Warren & Cooper, 2005). The common denominator in all cases is the ability of children to demonstrate algebraic thinking from early grades (Schliemann et al., 2003).

c) Difficulties and misconceptions

Research on junior high school learners’ understanding of algebra has resulted in records of difficulties and misconceptions around the use and interpretation of symbols, letters and variables (Booth, 1988; Herscovics & Lichevski, 1994; Stephens, 2005). Alibali et al. (2007) focussed on middle school learners’ understanding of algebraic concepts and found similar difficulties in relation to equality, variable, representational fluency and problem solving. Likewise, Asquith et al. (2007) discovered problems with equations and translation from verbal to

symbolic representations. Difficulties experienced by learners in the learning of algebra appear to indicate that certain conceptual changes are necessary to make the transition from arithmetic to algebraic thinking. Unfortunately one of the difficulties of transition from arithmetic to algebra in primary school is evident in learners' lack of understanding of equivalence and equations (Faulkner, Levi & Carpenter, 1999; Molina, Ambrose & Castro, 2004). Misconceptions about the equal sign leads to problems with learners' ability to solve and analyse equations and prevent them from learning arithmetic with understanding and developing a solid base for the future learning of algebra (Warren & Cooper, 2005, Molina & Ambrose, 2008). Children develop a limited concept of the equal sign while engaged in arithmetic learning and identify the symbol as a separation between what you start with from what you end up with. They do not contend with the notion that expressions on each side of the equals sign can be viewed as functions with variation and have difficulty recognising the diverse roles the equal sign serves in the case of formulas, identities, properties of numbers, and so on (Schliemann et al., 2007: xi-xii). Molina and Ambrose (2008) describe critical elements of instruction which deepen learners' conceptions about the equal sign and have found evidence that children begin to analyse expressions in ways that encourage algebraic thinking. Knuth et al. (2006: 8) acknowledge that if a goal of mathematics education is to help learners to better succeed in algebra then the "nature of learners" pre-algebraic experiences lays the foundation for more formal study of algebra.

2.3.3. The teaching perspective

There have been a number of international research projects focussing on early algebra and more specifically the role of the inservice teacher. These have occurred in different countries such as the USA, UK, Italy and Australia (Carpenter et al., 2003; Malara & Navara, 2004; Blanton & Kaput, 2005a, 2005b; Warren, 2008, Schifter, Russell & Bastable, 2009; Hunter, 2010). The research discussed within this section does not cover the full spectrum of what is available but it does highlight some of the key concerns and innovations that have occurred within the field of early algebra and teachers. Teachers fall within two categories, inservice and preservice, and research findings from each are discussed in some detail to help create an overview of the field of teacher education and its link to early algebra teaching. The results of empirical research in early algebra with inservice teachers, more especially the work of Blanton and Kaput, has also influenced the design of the Maths 2 course.

The first part of this section will look at teaching from the perspective of inservice professional development, starting with research from the USA and then moving on to work from Italy and Australia. The second part of this section will look at aspects of research within preservice teacher education and early algebra and establishes a rationale for the focus on preservice teacher development.

a) Inservice teaching perspective

There are many examples within the literature on early algebra which link teacher education, early algebra and professional development. Some of the projects will be described and various aspects will be highlighted that tell us more about how teachers are being prepared to develop and integrate algebraic thinking in their classrooms.

One of the earliest publications on teachers and algebraic thinking was *Fostering Algebraic Thinking – A Guide for teachers: Grades 6-10*, written by Driscoll (1999). This publication describes his perspective on algebraic thinking based on three different professional development programs with hundreds of teachers and learners. He identifies three essential algebraic “habits of mind”, which are at the core of algebraic thinking: Doing-Undoing, Building Rules to Represent Functions, and Abstracting from Computation. He then proposes a framework of questions for each of the “habits of mind” which teachers could use to develop learners’ algebraic thinking. He emphasises the need for teachers to explore the learning process from the learner’s perspective which necessitates attention to learning, balanced with attention to teaching in teachers’ professional development. Learners should be encouraged to work on problems in groups and discuss their learning so that productive habits of thinking are emphasised and associated with the effective understanding and use of algebra.

Carpenter et al. (2003, 2005) have also written extensively on developing algebraic thinking by integrating arithmetic and algebra in elementary school in a book called *Thinking Mathematically*, and in various reports and journal articles. They describe a “kind of mathematical thinking that can provide a foundation for learning algebra” which happens over a long period of time and starts in the earlier grades (Carpenter et al., 2003: 1). They have worked with teachers to study the development of learners’ relational thinking which can support their learning of algebra and to construct instructional approaches that support this development. The book

provides important guidelines for teachers on the teaching and learning of the big ideas of mathematics as well the mathematical ways of thinking that are involved in generating these ideas. They offer empirical evidence and examples of the tasks learners need to engage with and how they need to interact with these tasks.

One of the earlier teacher development projects in early algebra was “The Generalizing to Extend Arithmetic to Algebraic Reasoning Project” (GEAAR), a professional development program at the University of Massachusetts Dartmouth, designed to develop teachers’ ability to identify and strategically build upon childrens’ efforts to reason algebraically, and to use existing and supplementary resources to design classroom activities (Blanton & Kaput, 2005a). The format of the project required teachers to collectively solve authentic algebraic tasks, to adapt the tasks for classroom use, to implement these with their learners and to share their analyses of learners’ understanding with the group. Some of the research which emerged from this project has guided the development of algebraic thinking in the primary grades, by offering a guide to developing teachers’ “algebra eyes and ears” through the design of algebraic tasks, recognising and supporting learners’ algebraic thinking and creating a classroom culture that promotes algebraic thinking (Blanton & Kaput, 2003). There have also been detailed analyses into how and to what extent teachers are able to develop algebraic reasoning and this has led to a set of characteristics of the types of practices to support algebraic thinking and the coding of types of algebraic processes involved (Blanton & Kaput, 2005a). Results indicate teachers are able to plan and spontaneously support learners’ algebraic reasoning skills. This was supported effectively in the program through integrating the content and pedagogy of early algebra (Blanton, 2008).

The Supporting the Transition from Arithmetic to Algebraic Reasoning Project (STAAR) at the University of Wisconsin-Madison has also been involved in professional development of teachers. They have been active in helping teachers to recognise the potential of tasks to engage learner algebraic thinking, to recognise learner algebraic thinking and to develop thinking through question posing and task extension (Grandau & Stephens, 2006). The STAAR summer algebra institute (pilot professional development component of STAAR at the University of Colorado-Boulder) findings indicate the course has seen increases in teacher knowledge of algebra for teaching, increases in teachers’ pedagogical content knowledge and changes in instructional practices (Borko & Clarke, 2005a). They have identified

four strategies which appear to be fundamental in creating a mathematical discourse community in both professional development and classroom settings: posing rich tasks that promote discussion; establishing and maintaining a safe environment; asking teachers/learners to explain and justify their thinking; and encouraging teachers/learners to actively process each other's ideas (Borko et al., 2005b).

Greene (2004) and Johannig et al. (2009) offer suggestions to teachers on providing opportunities for understanding the big ideas in algebra through the selection of effective instructional materials and through opportunities presented by teachers. They recognise teachers have had limited experience in designing tasks that connect arithmetic and algebra and give suggestions for content and activities that teachers could use in the classroom.

The research of Malara (2004) and Warren (2008) looks at different aspects of professional development within early algebra, the former in Italy and the latter in Australia. The ArAl (Arithmetic Pathway to favour Pre-algebraic thinking) project is a “didactical innovation in mathematics, aimed at 5 -14 year olds, framed within the early algebra theoretical framework” (Malara & Navarra, 2008: 195). Its purpose is to design and implement a teaching sequence in which arithmetic and algebra thinking develop concurrently. Teachers play a critical role within the framework through critical reflection of their understanding of arithmetic and algebra and as agents of change within the classroom. Warren (2008) also looks at professional development with a focus on the development of a model to support the implementation of a new mathematical strand: Patterns and Algebra within the Queensland curriculum. The focus of her research is on the effectiveness of the model in assisting the teachers to understand and teach the content of the new strand. As a result of the professional development, teachers developed more confidence in mathematical knowledge and thinking within the strand pattern and algebra. The model's strength is in the interaction between the knowledgeable expert and the teachers and amongst the teachers themselves. Teachers were able to trial ideas and lessons, share children's work and strategies, and reflect on their practice. They grew in knowledge as they *co-constructed new knowledge* with an expert on the content and pedagogy of the strand.

In conclusion, there are several key findings from the research with inservice teachers, in the area of early algebra, which offer useful guidelines to teacher professional development in the future:

- Teachers need to be asking questions which will help develop learners algebraic thinking (Driscoll, 1999)
- Teachers need to be familiar with the big ideas in algebraic thinking and develop these concurrently with the development of arithmetic thinking to expose learners to the underlying structure of mathematics within arithmetic (Carpenter et al., 2003; Greene, 2004)
- Teachers need to select and design tasks which develop algebraic thinking. They need to create a culture which actively encourages learners to develop relational understanding and to recognise and encourage learners through questions and effective use of instructional materials (Blanton & Kaput, 2003; Grandau & Stephens, 2006; Johannig et al., 2009).
- Teachers need to be able to critically reflect on their knowledge of the teaching and learning of early algebra and this could be promoted through interaction between a knowledge expert and teachers and between teachers themselves (Malara, 2004; Warren, 2008).

b) Preservice teaching perspective

While much has been learned from research with teachers practising fulltime in classrooms, it is also imperative to examine the research within preservice primary school teacher education and early algebra, given the focus of this study on the development of preservice teacher content knowledge within a preservice algebra course.

Early algebra includes the work of developing algebraic thinking in the early grades (1 – 7). There are a variety of studies which have looked at preservice teachers and algebra in both primary and high school and these have been grouped into four categories: assessment of preservice teacher knowledge of algebra, development of preservice teacher knowledge of algebra, preservice teacher knowledge of algebra and learners, and lastly preservice teacher knowledge of patterns and functions. The last category is included to link to the focus of this study on the development of teacher knowledge of early algebra, specifically patterns and functions, and to examine some of the relevant literature in the field.

The first category deals with the assessment of preservice teachers' knowledge of algebraic thinking using different research methods. This is an attempt to gain understanding of preservice teachers' mathematical understanding of algebra to better develop the necessary knowledge and skills needed to teach algebra more effectively in the younger grades (Stump & Bishop, 2002; Van Dooren, Verschaffel & Onghena, 2002; Davis, 2005; Nillas, 2010). Stump and Bishop (2002) explored preservice and middle school teachers' conceptions of algebra using exemplary curriculum materials. They found teachers could do the mathematical tasks but could not explain their thinking and did not know the difference between algebraic reasoning and problem solving. They suggest that preservice teachers need to understand algebra as a way of thinking and a way of working with patterns that occur in everyday life and teacher education needs to organise teaching experiences which broaden preservice teachers' vision of algebraic so "they can promote algebraic reasoning in schools" (Stump, Bishop & Britton, 2003: 185).

Davis (2005) used a series of reflective questions based on the algebra courses taught in high school to examine the subject matter knowledge of preservice teachers. Results indicate the questions provided a useful opportunity to help student teachers to recognise their shortcomings and highlight the need to know more than the facts, terms and concepts of algebra but also the 'how' and 'why' they work. Similar results emerged from another study with preservice teachers based on algebraic tasks which had to be solved and alternative solutions provided (Haciomeroghi & Haciomeroghi, 2005). They found preservice teachers experienced problems in solving the tasks, had limited knowledge of the algebraic procedures they used and could not give meaningful explanations. Nillas (2010) investigated preservice teachers' mathematical understanding through the analysis of test items for classroom instruction involving linear, exponential and quadratic relationships. She found the items provided contexts for developing understanding and suggested that teachers need to engage in mathematical sense making and reasoning to experience what it means to teach and learn for understanding. It is imperative to have an experience of an environment which fosters this instructional goal and to encourage the development of a favourable disposition towards mathematics.

The research evidence in this category highlights the need for preservice teachers to not only know the mathematics they teach but also to develop the knowledge of

how to teach the concepts. This involves experiencing mathematics by understanding what they are doing, working with problems and learners' solutions, sharing strategies and engaging with algebra reasoning. They need to broaden their vision of algebra and to develop a disposition towards mathematics as a sense-making activity.

The second category of research development of preservice teachers' knowledge of algebra looked at different approaches to develop preservice teachers' algebraic content knowledge and attitude towards algebra using different methods: using an alternative teaching approach; analysis of preservice teachers' misconceptions; and performance assessments (Mc Gowen & Davis, 2001; Clements & Ellerton, 2008; Rangel-Chavez, Capraro & Capraro, 2008; Darley & Leopard, 2010). Some of the results from the examination of teacher knowledge discussed previously also touched on the strategies for the development of knowledge for teaching early algebra. Mc Gowen & Davis (2001) used an alternative approach to the training of teachers to enhance their thinking and their ability to see and value connections in algebra. Early algebra for them is "not only a list of topics or general aspects of mathematics" but "also an attitude to mathematical thinking" (p. 6). They propose a set of learning and teaching experiences, based on their research, that help students become effective teachers of early algebra through addressing attitudes to mathematics learning, seeing learners solve problems, seeking connections, hearing and seeing other insights, practicing interpretations, reading and discussing literature, adding and modifying existing knowledge and relating these experiences to learners.

Rangel-Chavez et al. (2008) also looked at the preparation of preservice teachers for algebra and designed an online module to develop algebraic habits of mind. The module allowed for collegial learning in professional learning communities through online discussions, chats and reflections, and activities included samples of children's work, videos, vignettes, and lesson planning. Their results indicate that preservice teachers can improve their algebraic habits of mind but they need lots of practice using a variety of activities. Clements and Ellerton (2008) used a five step teaching intervention focussing on errors and misconceptions to improve preservice teachers' algebraic content knowledge. The findings show that preservice teachers' algebraic content knowledge can be improved through identification of personal misconceptions, understanding where and why they occurred and revisiting relevant

tasks to build and consolidate understanding. Darley and Leopard (2010) used performance assessments (teaching demonstrations) to build preservice teachers' conceptual understanding and mathematical confidence for teaching arithmetic operations i.e. division of whole numbers, division of integers, division of fractions, and division of rational expression (e.g. $— \div 5$). The purpose of the assessments was to “build a foundation that will eventually strengthen learners' algebraic understanding” (p. 187) and although the assessments were time consuming, they were a “useful tool in strengthening the arithmetic-to-algebra connection” (p. 190).

Welder and Simonsen (2011) investigated the effects of an undergraduate mathematics content course for preservice primary school teachers. They measured the content knowledge of the teachers before and after the course and found significant gains in both common and specialised content knowledge in two prerequisite algebra concepts (numbers and equations/functions). This provides evidence that preservice teachers develop mathematical knowledge in “collegiate course settings” (p.1).

The results from the different interventions highlight the potential for teacher education to impact on the development of content knowledge for teaching early algebra. There are some key features which appear to emerge from more than one of the studies including: learning communities, collegiality, children's work, misconceptions, discussion and reflection, video, vignettes, solving problems, literature, relating experiences, and attitude. Preservice teachers need a set of learning and teaching experiences which enhances their thinking and ability to see and value connections in algebra. They need to understand their own errors and misconceptions, why and where they occur and how to rebuild and consolidate knowledge of algebra. This is often connected towards attitudes to mathematics learning and the ability to see and hear other insights. It involves reading and discussing relevant literature within the field of algebra teaching and learning in a professional collegial community. The use of video, vignettes of lessons, lesson planning and teaching demonstrations have been shown to help the development of preservice teacher algebra content knowledge.

The third category of research focussed on the link between preservice teachers' knowledge of algebraic thinking and learners. The research covers a number of different aspects such as interpreting learners' responses and written responses,

links between teacher algebraic thinking and questioning, and teacher ability to solve algebraic problems and to recognise and interpret algebraic thinking of learners (Von Dooren, 2002; Stephens, 2008; Van Den Kieboom & Magiera, 2010; Magiera et al., 2010). Research conducted with Flemish primary and secondary preservice teachers showed that 50% of the teachers had a lack of affinity for algebra and have great difficulty understanding and interpreting learners' solutions, and produced negative evaluations of the learners' work (Von Dooren, 2002). There is doubt as to whether these teachers had developed the disposition to prepare learners for the transition from arithmetic to an algebraic way of thinking.

Stephens (2008) examined conceptions of algebra held by 30 preservice teachers and found they were rather narrow and limited to a view of algebra as symbols and symbol manipulation. They also thought of algebra as a "method" or something one "does" as opposed to a way of thinking (p. 45). Their limited conceptions appeared to influence how they evaluated learners' work and the researchers suggest that what "counts" as algebra is what will be valued and encouraged in the classroom. Preservice teachers need to experience algebra as a way of thinking as opposed to a list of procedures to be followed if they are to move beyond their own limited conceptions to develop children's algebraic thinking. They are critical elements in the reform of algebra in primary school and the future of early algebra.

Other research on teachers and learners looked at the relationship between preservice teachers' algebraic thinking and their ability to ask questions. Researchers found that students with high levels of content knowledge ask more probing questions to engage learners in algebraic thinking while low algebraic thinking teachers asked more checklisting and prompting type questions (Van den Kieboom & Magiera, 2010). Magiera et al. (2010) highlighted what appears to be a gap in existing literature related to early algebra instruction and preservice teachers' knowledge of algebraic thinking. Using the same sample of preservice teachers as Van den Kieboom and Magiera (2010), they examined the relationship between the preservice middle school teachers' own ability to use algebraic thinking to solve problems and their ability to recognise and interpret the algebraic thinking of middle school learners. Results indicate the ability to recognise and interpret the algebraic thinking of learners strongly correlates with the teacher's own ability to use algebraic thinking to solve problems. They recommend that teacher education needs to build preservice teachers' knowledge of algebraic thinking, design programs that are

“sensitive to important issues related to early algebra instruction” and to strengthen “preservice teachers’ ability to apply the knowledge of algebraic thinking they learn in university coursework” (p. 33-34).

A summary of research on preservice teachers and learners clearly indicates the imperative and crucial role of knowing the mathematics content as well as how to use the knowledge in teaching. Preservice teachers who have a lack of affinity with algebra and poor ability to solve algebraic problems have great difficulty understanding and interpreting learners’ work. This can influence evaluation and produce negative feedback to learners. It can also impact on the levels of questioning in the classroom resulting in fewer probing questions and more checking and prompting questions. There is a need for teacher education courses which recognise these issues and remain mindful of the recommendations of the research.

The final category of this section looks at research related to preservice teachers and the topic of patterns and functions. Several researchers have heralded pattern explorations as a preferred introduction to algebra (Zazkis & Liljedahl, 2002), while others such as Kaput and Blanton (2001: 346) caution that generalising and formalising patterns and constraints is one of the forms of the “complex composite” of algebraic thinking.

Zazkis and Liljedahl (2002) explored the attempts of a group of preservice teachers to generalise a repeating visual number pattern and found a gap between these students’ ability to express generality verbally and their ability to employ algebraic notation. They were able to demonstrate algebraic thinking and could use algebraic notation but there was a lack of synchronicity between the two. Instead of seeing this as a problem, they suggest “this gap should be accepted and used as a venue for students to practice their algebraic thinking” (p. 400). The use of problems that are rich in patterns are not only good mathematical activities but they provide a valuable opportunity to appreciate the different ways of expressing generality. Sanchez and Llinares (2003) looked at four student teachers’ pedagogical reasoning on functions and found that students differed in their knowledge for teaching various aspects of the concept and in the use of representations to structure learning. They suggest that mathematics teacher education should approach subject matter knowledge with the purpose of teaching (pedagogical reasoning) in different ways

and create opportunities for students to “design learning tasks, analyse the mathematical field of these tasks and consider the curriculum learning goals ” (p. 23). They believe that mathematical knowledge in teaching should be a concern of teacher education and it is important to know more about the relationship between knowledge of the mathematical content and the use of knowledge in teaching to create better learning opportunities for learners.

A further three studies on preservice teachers and the topic of patterns and functions investigated the effects of interventions on preservice teacher ability to generalise and justify patterns. The first conducted a pretest/intervention/posttest study of a mathematics method course for preservice teachers using a problem solving based teaching approach (Hallagan et al., 2009). Their findings show that preservice teachers made significant growth in their understanding of algebraic generalisations as a result of the intervention activities but they found it easier to generalise for the $y = 4n$ type problem rather than the $y = n^2 + 1$ problem. Generalisation is a difficult topic but when problems are put into a context and taught in a problem solving environment, student teachers can show improvement. Another study used a teacher intervention to evaluate the preparation of preservice teachers to teach early algebra concepts with the goal of improving their ability to generalise and justify algebraic rules when using pattern-finding tasks (Richardson et al., 2009). They found students could initially generalise explicit rules using symbolic notation but had difficulty in providing justifications: however later results showed improvement in all but two of the justification abilities of students. The results of the intervention indicate the design and use of pattern finding tasks can be instrumental in mediating learning how to generalise and justify rules and there are observable patterns of growth in reasoning which provide a way to map preservice teachers' development of algebraic thinking over time.

Yesildere and Akkoc (2009) investigated the development of two Turkish preservice teachers' pedagogical content knowledge of learners' understanding of and difficulties with finding the rule of number patterns in a school practicum course. Their knowledge of number patterns improved through observation of number pattern lessons, discussion and analysis of learners' difficulties, and through the planning and teaching of pattern lessons. They also developed knowledge of content, curriculum and teaching approaches but used visual models to algebraic rules and privileged trial and error methods. They did not see good teaching

practicum examples of the effective use of visual models and believe that this reveals “the importance of the choice of teachers for the school practicum course” (p. 407).

The results of research on preservice teachers and the topic of patterns and functions suggests the inclusion of pattern activities in mathematics education courses provides a good opportunity for student teachers to appreciate different ways of expressing generality. The design and use of pattern tasks can be instrumental in mediating learning how to generalise and justify rules. Although there is acknowledgement of the difficulty of generalisation tasks, not too dissimilar to the results from learner research, when problems are put in context within a problem solving environment, there is improvement in the understanding of algebraic generalisation. There is also research evidence to suggest mathematics education courses which include opportunity to design pattern tasks, to analyse the mathematics of the task and to link the task to the curriculum, help student teachers to connect knowledge of mathematical content and how to use the knowledge in teaching. Preservice teachers’ knowledge of number patterns also improves through observation of number pattern lessons, discussion and analysis of learners’ difficulties and the planning and teaching of pattern lessons.

All of these suggestions provide a need and justification for this study on the development of preservice teachers’ knowledge to teach early algebra. It is imperative for teacher education to be active and reflect on the nature of how knowledge for teaching early algebra develops.

In conclusion, there are several recommendations from the empirical evidence regarding preservice teachers and early algebra which relate to features of this study. The design of a course to develop preservice teachers’ knowledge for teaching early algebra needs to be sensitive to knowledge and conceptions student teachers bring to the course. They need to have experiences which enhance their algebraic thinking and help make connections. The pedagogy of the course should model teaching for understanding and focus on developing connections between different knowledge domains. It needs to include building preservice teachers’ knowledge of algebra and early algebra, reading, discussing and critiquing relevant literature, integrating and demonstrating different approaches to teaching algebraic thinking, making use of videos, vignettes, lesson planning and demonstrations and recognising learner errors and misconceptions. Patterns and functions have been

identified as an important route to formal algebra and provide the content focus for this study.

2.3.4. Early algebra and curriculum

Curricula have a significant influence on what happens in teaching and learning and ultimately on the mathematical performance of learners. While there is continued debate on the nature of algebraic thinking, there is evidence of the introduction of algebraic ideas at various levels of the curriculum in many different countries. Ferrucci (2004: 131) gives an overview of a series of articles dealing with “differing perspectives” and the “scope and sequences of effective activities that develop algebraic thinking skills” across the curricula of China, Singapore, Korea and the USA. She presents each country’s case and discusses the differing perspectives on when algebraic thinking skills should be introduced into the primary grades (Ferrucci, 2004).

While algebraic thinking is not explicitly mentioned in the Singapore curriculum; there is evidence of activities which contribute to its development throughout the primary grades. Korean learners only begin the formal study of algebra in Grade 7, but there is evidence of some algebraic activities within the early primary school grades. The Chinese curriculum includes algebraic tasks in the first four grades and the formal study of algebra begins in Grade 5. The USA curriculum includes a comparison of two different programs, the first *Investigations* which starts the development of algebraic thinking skills in Grade 1, while the alternate Davydov curriculum starts with the study of algebra (concepts of equality and other algebraic notions) before the study of arithmetic. Overall, it would appear, there is great variation across cultures as to what constitutes algebraic thinking at the primary school level and when this is best introduced. However, research shows that learner success in high school algebra is clearly dependent on “quality mathematical experiences” which are needed to “develop algebraic readiness” (Ferrucci, 2004: 138).

Cai and Knuth (2005:1) provide a summary of algebraic thinking in the USA and some Asian countries which gives a broader view of “curricular approaches to integrating algebraic ideas into earlier grades” and “provides insights regarding the development of students”. They edited a special journal publication on the topic of algebraic thinking and curricula and looked across a set of articles to highlight two

important themes. The first of these is the need to integrate better opportunities to develop learners' algebraic thinking through making connections between arithmetic and algebra at the primary school level and to extend these connections through to high school. The second is the need to support teachers' work to implement practices that foster the development of learners' algebraic thinking (p. 3). If we want to prepare learners that are better prepared for the formal study of algebra in later grades then we also need to better prepare our teachers for the task.

A detailed analysis of the development of algebraic thinking in the South African curriculum was conducted by Vermeulen (2007) and later used by Roberts (2010) to compare the treatment of algebra content in the intended curriculum, for the early grades, in England and South Africa. The comparison was conducted on two levels: firstly the content structure of each curriculum and secondly a contrast of the detailed learning objectives (England) and the assessment standards (South Africa) given in algebra. Results indicate the South Africa intended curriculum showed commitment to the teaching and learning of algebraic thinking in the early grades but lacked substance while the English intended curriculum includes more algebra content but is not explicit about the development of algebraic thinking in the early grades i.e. generalised arithmetic and functional thinking. This research stresses the need for more detailed guidance for South African teachers on the treatment of early algebra, and the need for England to simplify and streamline the supporting curricular documentation to make early algebra more explicit (p. 173). Roberts (2010: 173) hopes the analysis will encourage "reflection on how intended curricular may be used to better communicate research in early algebra, at an appropriate level of detail, to teachers in the early grades".

The conclusion to be drawn from the research on early algebra and the curriculum is that there is variation across cultures about what constitutes algebraic thinking and when best to start teaching algebraic thinking. There is an identified need to integrate opportunities for learners to develop algebraic thinking in the early grades and to extend this to high school. Research from a number of different countries and contexts emphasise the need to support teachers to foster learners' algebraic thinking and to better prepare teachers for the task through detailed guidance in the treatment of early algebra. Preservice teacher education courses can help support and give guidance to teachers for the development of algebraic thinking both for themselves and with their learners. It is important to give substance to preservice

teachers' understanding of early algebra and its link to curricula through working with the curriculum in the design and implementation of early algebra lessons. They need to develop connections between their knowledge of content and their knowledge of the curriculum for early algebra.

2.4. Conclusion and theoretical framework

This literature review addressed two important aspects within this study into understanding the development of preservice teachers' knowledge for teaching early algebra, namely teacher knowledge and early algebra. The first aspect of this review starts with the seminal work of Shulman into teacher content knowledge: subject matter knowledge, curricular knowledge and pedagogical content knowledge and linked this to subsequent research in teacher knowledge within the field of mathematics education (Adler & Davis, 2006; Ball et al., 2008; Davis & Renert, 2009; Rowland et al., 2009, Watson & Barton, 2011). There was a detailed overview of the different perspectives of teacher knowledge and mathematics, outlining the main ideas and distinctions between each approach. This section included an extended discussion of the mathematical knowledge for teaching model (Ball et al, 2008), its history and development, including explanation of the domains of knowledge, links to research and results from use of the model in mathematics education. There is further discussion of the role of the model in the design of the Maths 2 course and the analysis of the data in Chapter 3.

The second aspect of this chapter was early algebra. It looked at the development of early algebra as a focus of study in mathematics education and included a working definition of early algebra as it is used in this study. There was a summary of some of the empirical research in early algebra with learners and teachers and how different countries integrate early algebra in their curriculum. There were a number of themes which emerged from research and resonated throughout the literature in terms of understanding what early algebra means, how best to educate teachers and learners to support the development of algebraic thinking and how curriculum guides this process. The research work of Carpenter et al (2003); Blanton & Kaput (2005a, 2005b); Carraher et al. (2006); Blanton (2008); and Mason (2008) all pointed to the vital role of the teacher in developing algebraic thinking through the planning of suitable lessons and the act of teaching. However, the call for the reconceptualisation of algebra in the primary school places a great demand

on teachers who have had insufficient training in the new approach (Stephens, 2008).

This study has been designed to understand the development of preservice teachers' knowledge for teaching early algebra. It is important for preservice teacher education to be cognisant of current, local and international thinking in the teaching and learning of early algebra in order to understand the knowledge that will be required by teachers to teach this topic effectively. It is also imperative to determine what learning experiences assist the development of this knowledge.

Understanding the development of preservice teachers' knowledge for teaching will focus specifically on subject matter knowledge which includes common content knowledge and specialised content knowledge for teaching early algebra (Ball et al., 2008). The methodology involved in this study will be described in detail in Chapter 3.

CHAPTER 3. RESEARCH DESIGN AND METHODOLOGY

3.1. Introduction

This chapter outlines the research design and methodology of the study. It begins with a discussion of the three types of research design which are most prevalent in social sciences research and locates this study within one of these. This is followed by an explanation of the philosophical worldview (paradigm) which underpins this study, as well as the strategy of inquiry and the research methods employed to collect the data. The data analysis process is outlined in detail, including issues of reliability, validity and generalisability.

3.2. Review of the purpose of the research

This study investigates the development of knowledge for teaching early algebra in the context of primary preservice teacher education. Algebra is a notoriously difficult subject for most high schools learners and often impacts adversely on their mathematical progress. Negative attitudes towards the subject are then developed and entrenched and are carried through into teacher education with dire consequences for subsequent mathematics teaching and learning. A mathematics education course (Maths 2) was designed by me, as part of a Bachelor of Education programme, to address issues of content and pedagogy in the teaching and learning of early algebra, and most importantly, to develop preservice teachers' knowledge for teaching early algebra. It aimed to develop preservice teachers' perceptions of algebra from a formal system of manipulation and symbolisation towards a more holistic understanding of algebraic thinking in the primary school.

3.3. Research design

There are three types of research designs namely quantitative, qualitative and mixed methods. Although they can be represented as discrete approaches, they are better perceived as different positions on a continuum. An over-simplification of the distinction between the different approaches is to frame qualitative research in terms of words and quantitative research in numbers. Bryman (2008: 21-22) perceives the difference as deeper than the superficial issue of "presence or absence of quantification". He understands quantitative and qualitative research as different approaches to social research in terms of the connection between theory and research, epistemology and ontology, as explained below.

Quantitative research focuses on testing theories by examining the relationship among variables to explain human behaviour. It is informed by a positivist perspective which sees social reality as an external, objective reality. Qualitative research focuses on exploring and understanding the “meaning individuals or groups ascribe to a social or human problem” (Creswell, 2009: 4). The world is not ‘out there’ and separate from those involved in its construction. People are creators of their “social world rather than passive objects” (Bryman, 2008: 34). Mixed methods research is an approach to inquiry that combines both qualitative and quantitative research approaches and methods to provide a more comprehensive study of an inquiry.

There are a variety of frameworks for research design which classify and explain theory in different ways. Creswell (2009: 5) provides a useful interconnected framework for research design which comprises three elements: philosophical worldviews (paradigm), strategies of inquiry and research methods. The nature of these elements varies depending on the nature and purpose of the research. This is illustrated in Figure 3.1 (Creswell, 2009: 5):

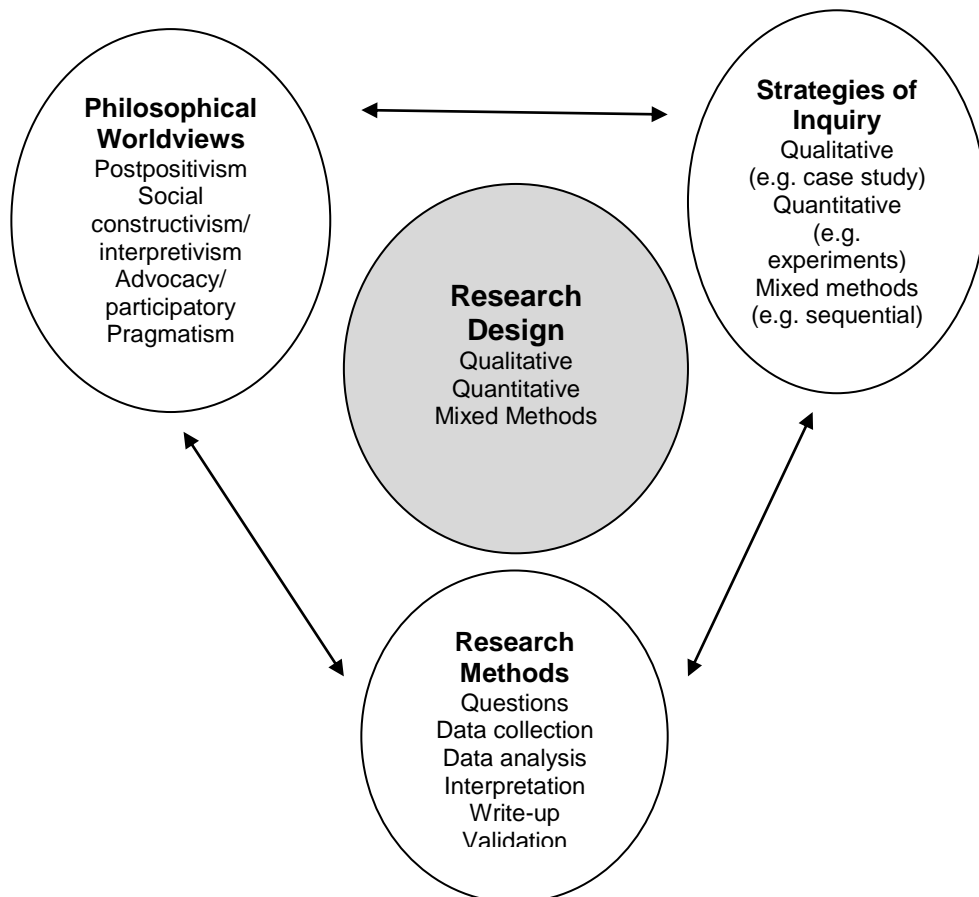


Figure 3.1: Interconnected framework for research

Creswell's use of the term worldview (paradigm) is defined as the set of beliefs that guide action, and includes the general orientation about the world and the nature of research held by the researcher. He focuses on four worldviews (paradigms): postpositivism, social constructivism (often combined with interpretivism), advocacy/participatory and pragmatism. Each of the worldviews is linked to either a qualitative, quantitative or mixed methods approach which employs particular strategies of inquiry and data collection.

The study reported here is a qualitative study within an interpretivist paradigm in that it is interested in understanding the meanings that preservice teachers ascribe to their experiences of the Maths 2 course and teaching practicum. These meanings are then used to understand the development of knowledge for teaching early algebra for this group of preservice teachers.

Babbie & Mouton (2011: 33) identify four themes which are important when electing to use an interpretivist approach to research design. Firstly, interpretivism is a theory of knowledge (epistemology) which involves understanding human actions in terms of the meanings, intentions, values and beliefs that people hold. This means that the data collection methods need not only capture the observable but also provide insight into the phenomena from different perspectives. Secondly, interpretivism emphasises the subjective role of the researcher and the need to get close to the perspective of the insider as opposed to the natural science attitude of keeping a distance between the researcher and the object of study. Thirdly, there is an emphasis on understanding rather than objective explanations which allows interpretation of the "social actors themselves". Lastly, there are methodological implications when using an interpretative approach to understand the "meanings and self-descriptions" of the individuals in the study. This impacts on the types of data collection used as well as the data analysis processes needed to understand the individuals in terms of their own interpretations of reality and understanding of society. Each of these four themes is elaborated in detail in the research methodology, methods and analysis sections that follow. There is also some discussion of the limitations of this approach as they apply to this study.

3.4. Research methodology

The purpose of this study was to understand the development of preservice teachers' knowledge for teaching early algebra. The participants were a group of

third year Bachelor of Education preservice primary school teachers taking a Maths 2 course and teaching practicum to build knowledge for teaching early algebra. There are different qualitative research methodologies which offer different strategies of inquiry such as ethnography, grounded theory, case studies, phenomenological research and narrative research. This study employed a case study approach as it facilitated exploration of the development of knowledge for teaching early algebra within the context of preservice teacher education, using a variety of different collection procedures (Creswell, 2009). It allowed for the exploration, through many different lenses, and understanding of multiple aspects of the phenomenon (Baxter & Jack, 2008).

The remainder of this section outlines the purpose of case studies, different approaches to case study methodology; the role of theory in case studies; measures of quality; strengths and limitations of case study and the role of the researcher.

3.4.1. Purpose of a case study

Given that the primary purpose of a case study is to allow for an in-depth investigation of simple and complex situations using a variety of data sources, it also helps to answer the how and why questions that can be used to explore the uniqueness of a case (Stake, 1995; Simons, 2009). Harling (2002) describes a case study as a holistic inquiry used to investigate a contemporary phenomenon within its natural setting. Yin (2003: 23) defines case study research as an “empirical inquiry that investigates a contemporary phenomenon within its real-life context; when the boundaries between phenomenon and context are not clearly evident; and in which multiple sources of evidence are used “. The phenomenon/case in this study was to understand the development of knowledge for teaching early algebra within the context of the preservice teacher education Maths 2 course. It involved multiple sources of data collection in the form of video recordings, lesson reflections, video questionnaires and focus group interviews.

Rule & John (2011: 7) identify five possible reasons for choosing a case study approach. Firstly it provides thick, rich descriptions of the case which help generate an understanding and insight into a particular issue. Secondly, it allows for exploration of a problem within a limited and focussed setting. Thirdly, it can help generate theoretical insights within the phenomenon or develop and test existing theory. Fourthly, a case study can help in the understanding of similar cases thus

providing a level of generalisability. Lastly it can be used in teaching to illuminate broader theoretical and/or contextual issues.

Case study methodology encourages close collaboration between the researcher and the participants to gain a better understanding of the participants' view of reality and to better understand their actions (Baxter & Jack, 2008). It requires the researcher to interpret events and to obtain a detailed insight into the research from different perspectives. This study took place in 2010 while preservice teachers were taking the Maths 2 course which aimed to develop their knowledge for teaching early algebra. As the mathematics educator, I was both teacher and researcher, which allowed for comprehensive investigation within the setting to understand the development of their knowledge for teaching early algebra over a year long period. The study took place over a year long period during which time I was able to collect data from a variety of sources: lesson reflections; questionnaires; focus group interviews and video recordings to provide rich and in-depth accounts of the phenomenon and context.

While there are many important reasons for using case study as a research methodology, there are those who view it as less desirable than other forms of empirical inquiry. Yin (2009: 14-15) highlights four traditional prejudices against the case study method: lack of rigor; it provides little basis for scientific generalisation; it takes too long and produces massive documentation and is too often used to establish causal relationships in randomised field trials. While Yin acknowledges the many problems in using case studies, he argues that each of these prejudices can be overcome through concerted efforts. He suggests that researchers using case study methodology should "work hard to report all evidence fairly", and case studies, "like experiments, are generalisable to theoretical propositions and not to populations or universes" (p. 15). He argues that case studies in the past may have been lengthy and unreadable but they do not solely depend on ethnographic and participant-observer data and can be valid and high-quality pieces of research. According to Yin (2009: 18), there is no doubt that good case studies are difficult to conduct. However a case study is an empirical inquiry that:

- investigates a contemporary phenomenon in depth and within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident

- copes with a technically distinctive situation in which there will be many more variables of interest than data points
- relies on multiple sources of evidence, with data needing to converge in a triangulation fashion
- benefits from the prior development of theoretical propositions to guide data collection and analysis.

The reason I selected a case study methodology was to provide an “intensive, holistic description and analysis” of the development of preservice teachers’ knowledge for teaching early algebra in the primary school (Merriam 1988: 16). As an interpretive researcher, it was important for me to have opportunities to study the meanings that preservice teachers construct in the learning and teaching of early algebra and to proceed from these meanings to make sense of knowledge for teaching early algebra (Thomas, 2011).

3.4.2. Types of case studies

While there are many different types of case studies, two key approaches have directed case study methodology over the past decade, the one developed by Stake (1995) and the other by Yin (2003, 2009). They are both interested in the exploration of a phenomenon within its context but they each use different methods. Stake (1995: 3) contrasts three types of case studies: intrinsic, instrumental and collective. An intrinsic case study is used when a case is unique or of particular interest in itself, or when there is an interest in learning more about the particular case. The researcher has to be able to define the uniqueness of this phenomenon which distinguishes it from all others. An instrumental case study is conducted to explore a broader issue using a particular case; it provides insight into a particular issue and/or helps to refine a theory. The case is often looked at in depth, its contexts scrutinised and its ordinary activities detailed (Baxter & Jack, 2008). The case can be typical or unusual and does not depend on the researcher defending its typicality but does have a rationale for using the particular case (Harling, 2002). Collective case studies involve the use of several cases to form a collective understanding of the issue or question (Stake, 1995: 4).

Yin (2003) provides an alternative classification of case studies based on their purpose: explanatory, exploratory, and descriptive. He differentiates between single case studies, holistic case studies and multiple case studies. An explanatory case

study seeks to explain what happens in a particular case, why it happens and often tests existing theory or generates new theory. An exploratory case study is used to explore a situation in which the intervention being evaluated has no clear set of outcomes. It can also be used to examine a phenomenon that has not been investigated before and could possibly lead to further research. A descriptive case study is used to describe an intervention or phenomenon and the real-life context in which it occurred (Yin, 2003).

This research involves an instrumental, descriptive case study which tries to describe and understand the development of a group of preservice teachers' knowledge for teaching early algebra through the use of verbal and written responses and video lesson recordings. The responses and video recordings are used as illustrations to provide insights into the development of knowledge for teaching as it happens during the Maths 2 course. It acknowledges that knowledge for teaching is constantly transforming and is not static.

3.4.3. Theory and case study research

Harling (2002: 3) highlights the place of theory in case study research and explains that it can be different depending on the nature and stage of the investigation. Theory can be absent from studies when the focus is on describing the case and its issues; alternatively, theory in case studies can be used to modify generalisations (Stake, 1995). Yin (2009: 35) believes that theory development is crucial to the design phase of a study regardless of whether the purpose is to “develop or test theory”. Creswell (2009) suggests another perspective in which theory becomes the end point which involves an “inductive process building from the data to broad themes to a generalised model or theory” (Creswell, 2009: 63).

Rule & John (2011: 96 - 100) highlight three different approaches to theory and case studies: firstly the deductive, theory-verification design; secondly the inductive, theory-building design and thirdly the dialogical design which involves interaction between theory and the case. In the first instance, deductive theory starts with a theory moving from the general to the specific and is prevalent in explanatory and instrumental case studies which start with a theory. It is useful for testing and developing theories in new contexts. The second approach is inductive and linked to the generation of theory moving from the specific to the general. It is often used in exploratory case studies which focus on specific instances and tends to make

tentative generalisations for future research. This approach to theory is usually found in grounded theory and phenomenology. It attempts to generate theory from the ground up and avoid imposing theory from above. The third approach of understanding theory and research within a case study entails a shift away from a linear progression between theory and case towards a recursive process. It involves a dialogical approach which is a move backwards and forwards between theory and the practice of research (case) at different stages in the research process. Theory can be used to inform research at all stages in the process and alternatively the practice of research (case) can help develop, modify and revise theory.

This study employed a dialogical approach in which theory was used initially as a theoretical lens to provide an “overall orientating lens” for the study of knowledge for teaching early algebra (Creswell, 2009: 62). The theory helped shape the research questions, the design of the mathematics education course and the analysis of the preservice teachers’ knowledge for teaching early algebra. It provided a critical foundation for the design of the study both as a way of structuring the research as well as developing the theory in the context of preservice teacher education. The literature review helped clarify the theoretical framework for the study, the focus and purpose of the study as well as the collection of data. The analysis and interpretation of the data required a recursive process in which it was necessary to move constantly between the theory, the case and the data to provide a “systematic and in-depth investigation of a particular instance in its context” in order to better understand how knowledge for teaching early algebra develops (Rule & John, 2011: 4). The theory was used and developed to filter and organise the findings without excluding the need to be sensitive to paradoxes within the case and to pursue them when necessary (Harling, 2002). It is best demonstrated in the following way (Rule & John, 2011: 100):

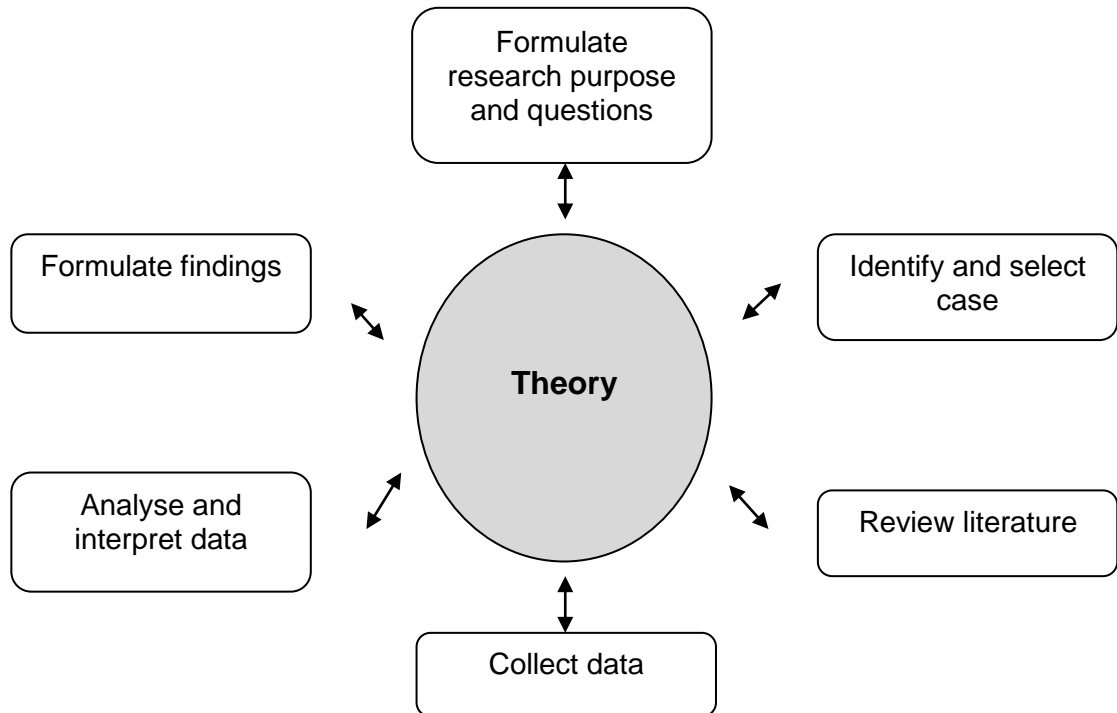


Figure 3.2: A dialogical approach to theory in case study

3.4.4. Measures of quality

There are three measures of quality which feature prominently in both quantitative and qualitative research, namely *validity*, *reliability* and *generalisability*. Rule & John (2011: 104) describe the three measures within a quantitative tradition such that *validity* is seen as the measure which allows the researcher to make “claims about the focus or phenomenon being studied”, *reliability* involves the “replicability of the findings” and *generalisation* is related to “notions of prediction” for a larger population. Researchers working within the qualitative tradition have “attempted to ensure quality through the development of alternative measures and processes” (p. 106). Although there is no consensus on issues of reliability, replicability and validity, there has been a lot of discussion on “external validity or generalisability of case study research” (Bryman, 2008: 55). Case study researchers do not delude themselves that it is possible to identify typical cases that can be used to represent certain classes of objects. It is rather an intensive examination of a single case which allows for both theory generation and theory testing. This does not limit the research to an inductive approach but rather encourages the researcher to work within deductive and inductive approaches (Bryman, 2008).

Guba (1981) offers an alternative measure of quality which is a shift away from the traditional conceptions of validity and reliability towards the concept of *trustworthiness* (in Rule & John, 2011: 107). This involves giving attention in the study to *transferability, credibility, dependability and confirmability*. The idea of *transferability* is re-defined as an alternative to external validity or generalisability through the provision of thick descriptions of the case and its context by the researcher including findings and conclusions. The reader can then decide if the case is believable and whether it resonates with other familiar cases. This allows for “opportunity for the particular to intersect with the general” i.e. reader-determined transferability (p.105). *Credibility*, an alternative to internal validity, is the extent to which the case study has been described and explained. *Dependability* focuses on the “methodological rigour and coherence in generating the findings and the case accounts” which can be accepted with confidence. *Confirmability* means acknowledging and “addressing concerns about the influences and biases” of the researcher (p. 107).

Case study research involves multiple data sources to allow for intensive examination of a setting or ‘case’ (Bryman, 2008: 53). This enhances data credibility and can be derived from, but is not limited to, observation, interviews, documents and artefacts. Qualitative data can be collected and integrated to present a holistic understanding of the case/phenomena being studied. Baxter & Jack (2008: 554) propose that each source of data should be seen as a piece of the puzzle which contributes to the overall understanding of the phenomenon and where the data converges. Thus a stronger and greater understanding of the case emerges.

Yin (2009: 116) suggests that the process of *triangulation*, which involves using “several different sources of information” to support the findings, helps provide a more “convincing and accurate” account of the case study (i.e. strengthen validity and improve quality). Henning et al. (2004) suggest an alternative route to quality through the process of *crystallisation* (in Rule & John, 2011: 109) in which “additional sources and methods show up additional facets rather than confirming some true position, as signalled in triangulation”.

This study addresses issues of *trustworthiness* by providing rich, thick descriptions of the case study, findings and conclusions to help the reader decide if they are

transferable to other or similar contexts (*transferability*). There is a detailed explanation of the case study including multiple sources of data (interviews, video observations and document analysis) as well as data analysis to enhance the *credibility* of the study. The *dependability* of the findings is demonstrated in the description and coherence of the methodological process and the concerns about the influences and biases are outlined from the beginning of the process (*confirmability*).

Practical steps to improve the trustworthiness of this case study included the involvement of other mathematics educators in the analysis phase to check coding and to give feedback to enhance the dependability of the study (Baxter & Jack, 2008).

3.4.5. Strengths and potential limitations of case study research

Harling (2002) suggests that in order to understand a phenomenon, especially where humans are involved, all aspects of the situation have to be considered which makes the case unique. This means that the researcher has to consider many variables within the case (s) and needs to probe deeply into a situation. According to Simons (2009) a case study can document multiple perspectives, explore contested viewpoints and demonstrate the influence of key participants in a study. It can help to describe, document and interpret events as they unfold in a 'real life' setting and is flexible and not constrained by time or methods. Case studies have the potential to engage the participants in the research process and can be accessibly written to allow the reader to experience what has been observed and help build tacit understanding of the case. It can also provide a context for researchers to take a "self reflexive understanding of the case and themselves" (p.23).

Yin (2009) acknowledges the concerns and cautions researchers using the case approach to work hard to report all evidence fairly. Indeed, case study reports are often too long and detailed to read and sometimes include descriptions that are overly convincing. While many of the concerns of case study research need to be appreciated and recognised they do not necessarily have to be seen as limitations. The purpose of a case study is not to formulate generalisations and proof of a case or to test a theory but rather to understand the particular, to present a rich portrayal

of the case, to establish the value of the case and/or to add to knowledge of the specific case (Simons, 2009).

3.4.6. Role of the researcher in case study research

I was an active co-participant in the research in that I was delivering the mathematics education course as well as researching the development of the preservice teachers' knowledge for teaching early algebra. I tried to manage the issue of bias by using a variety of data collection methods in order to provide alternate perspectives of the same concept (data crystallisation). The subjectivity of the researcher is an inevitable part of case study methodology and while this could be seen as a weakness, it gave me, the researcher, a more detailed and intimate knowledge and experience of what was happening in the lectures as well as the learner classrooms while the preservice teachers presented their lessons (Simons, 2009). The preservice teachers were encouraged to engage and reflect on their knowledge and practice throughout the year and willingly participated in the program. I was able to research with them and not simply gather data about them.

The purpose of the research study was explained at the beginning of the academic year to the preservice teachers registered for Maths 2 and they were given the choice to participate in the study or not. All preservice teachers agreed to take part in the research and signed consent forms to confirm their involvement with the assurance of confidentiality in all distribution of the findings. They agreed to have their early algebra lessons videoed and were each given their own personal copy of the videoed lesson to keep.

The issue of subjectivity also arose in the interpretation of the preservice teachers' responses and actions and in the selection of illustrations used as manifestations of knowledge for teaching early algebra. My explanation of the data analysis process and findings contains detailed accounts of the process I followed and the decisions I had to make. As an active participant in the research as teacher educator and researcher, it was imperative to constantly reflect on my decisions and to account for this in the writing up process.

3.5. Research methods

The third element of the Creswell (2009) framework for design refers to the specific research methods used in a study and include the forms of data collection, analysis

and interpretation of the data. I start this section with an overview of issues related to the selection of the research site and sample group as well as the design and implementation of the Maths 2 course used to develop preservice teachers' knowledge for teaching early algebra. The Maths 2 course is a crucial element in the development of knowledge for teaching early algebra and is discussed in some detail to provide an in-depth overview of the purpose, structure and content of the course and to support *dependability* by illustrating the rigour and coherence of the design (Rule & John, 2011). This is then followed by an explanation of the data collection methods within this case study and a description of the data analysis process with some of the preliminary analytical categories.

3.5.1. Site

The site selected is a university in the Western Cape which is the biggest provider of primary school educators in the province. It has also been my workplace since 1996 in the field of mathematics education. The university offers a Bachelor of Education degree for the General Education and Training band (GET) at the Foundation Phase (Grades R-3) or the Intermediate/Senior Phase (Grades 4-9) level.

The structure of the four year BEd degree includes a number of compulsory courses with a special emphasis on language and mathematical development. Preservice teachers from both the Foundation Phase (FP) and Intermediate/Senior Phase (ISP) have to take compulsory mathematics education courses for the first two years of their degree (Introduction to Mathematics and Mathematics 1 respectively). Thereafter, mathematics education becomes an elective at the third and fourth year level. This study was conducted with a combined group of preservice teachers from FP and ISP who had elected to take the specialist Maths 2 course in the third year of their degree.

3.5.2. Sample

The subjects of this study were an opportunity sample of twenty six, third year Bachelor of Education - GET preservice teachers who had elected to take the third year mathematics specialisation course. Mathematics education courses offered in year one and two (Introduction to Mathematics and Mathematics 1) of the BEd degree focus on the learning and teaching of whole and rational number, patterns, geometry, measurement and data handling and included both content knowledge and pedagogical content knowledge for teaching the different topics. This sample of

preservice teachers registered for a Maths 2 course with a specific focus on algebra development from early algebra to more formal school algebra. The participants included 9 FP preservice teachers (all female) and seventeen ISP preservice teachers (4 male and 13 female), aged between 20 and 25 years old. They had a strong mathematical history starting with their Grade 12 results and carrying through the first and second year of their degree. They were all English first language speakers except for two Afrikaans speakers. The course catered for both FP and ISP preservice teachers working together and was designed to highlight the progression and development of algebraic concepts across the GET phase.

Maths 2 catered for preservice teachers who wanted to become mathematics specialist teachers in the primary school. As mentioned previously, it was an elective course and preservice teachers had to successfully complete first and second year mathematics education courses with more than a 60% average to gain access to the course. Maths 2 was a full year course which comprised of 24 academic weeks and an additional 8 weeks of teaching practicum in school. Lectures were 45 minutes long and took place three times per week. The course was made up of three different modules: Algebra (70%), Measurement (20%) and Data Handling (10%).

3.5.3. Conceptual underpinnings and design of the Maths 2 course

The purpose of the Maths 2 course was to equip preservice teachers with the necessary knowledge, understanding and skills to teach mathematics competently and confidently in the GET band. The conceptual framework of the course was located within a situated perspective of teacher learning and was guided by two principles: (1) enhancing and developing knowledge for teaching mathematics and (2) the central role of community in fostering teacher learning. Within a situated view of learning, knowledge is produced by interaction among people and is not communicated but rather constructed by individuals operating within a social context (Garcia, Sanchez & Escudero, 2007). Knowing is viewed as the practices of a community and the abilities of individuals to participate in those practices; learning is a social process that takes place in a participative framework and not isolated in an individual's mind (Tirosh and Even, 2007). Learning is strengthened within, for example, the practice of shared mathematical activities. This approach views learning as an active construction of meaning by making sense of mathematics problems, "combined with the development of social norms and practices, such as

inquiry, reasoning, explanation, justification, argumentation, and intellectual autonomy” (Even, 2005: 350). Just as children learn as a process of both construction and enculturation, so too a teacher learns a particular set of knowledge and skills, and the situations in which a teacher learns are fundamental to what is learned (Borko et al, 2005b).

The concept of situated learning rests on the notion of communities of practice as an important ingredient for teacher learning. Wenger (1998, 2006:2) identifies communities of practice as “people who engage in a process of collective learning in a shared domain of human endeavour”. However he cautions that not everything called a community is a community of practice and highlights three crucial characteristics: domain, community and practice. Although preservice teachers may not initially belong to the ‘community of practice’ of mathematics teachers, mathematics teacher education programmes need to provide the means to qualify them for becoming members of that community (Garcia, Sanchez & Escudero, 2003). While they are engaged in the activity of constructing knowledge and skills that are needed in the teaching of mathematics, they become integrated into the community of mathematics teachers through becoming communities of learners.

The preservice student teacher community of practice has an identity defined by a shared *domain* of interest: learning to teach mathematics. The members pursue their interest in the domain (learning to teach mathematics), engage in joint activities and discussions, help each other and share information. Through their participation in class activities and through their planning of lessons, they build relationships that enable them to learn from one another. However a community of practice is not just a community of interest. The Maths 2 course requires preservice teachers to interact with each other, explain their ideas and solutions, evaluate ideas, challenge thinking, as well as formulate and argue their understanding of mathematical problems. The norms of the discourse also emphasise “respectful attention to others” opinions and efforts in order to reach mutual understandings based on mathematical reasoning (Even, 2005: 351). It is crucial that preservice mathematics education courses are an essential part of the process of introducing teachers into the community of practice of mathematics teachers (Garcia et al., 2003).

a) Knowledge for teaching mathematics

The primary goal of the Maths 2 course was to develop new knowledge and ways of understanding and acting both in the area of mathematics and mathematics teaching with a focus on early algebra. It was about knowing and understanding “what teachers need to know” and “know how to do” so they can deal with learners’ responses and in ways that produce mathematical proficiency (Adler, 2005). Mathematical proficiency includes having conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and a productive disposition towards mathematics (Kilpatrick et al., 2001). The Maths 2 course was focused specifically on the development of knowledge for teaching algebra in the GET band. It addressed different domains of knowledge from the MKfT model with a particular emphasis on common content knowledge (CCK) and specialised content knowledge (SCK) for teaching early algebra.

The CCK component of the Maths 2 course dealt with knowledge of key algebraic concepts such as variable, equality, pattern recognition, representational fluency, functional reasoning, systems of equations, linearity and non-linearity, slope (proportional concept), continuity (discrete vs continuous), scale, and intercepts (see Appendix A: subject guide). The SCK aspects of the Maths 2 course linked theoretical perspectives in early algebra through the reading and discussion of current theory and research articles, including reference to the South African revised national curriculum statement (DoE, 2002). SCK is the term used for the mathematical knowledge and skills unique to teaching and though common content knowledge is clearly an essential component of teacher knowledge, it is also generally accepted that it is insufficient on its own to prepare teachers for the teaching and learning of mathematics (Ball et al., 2008; Ma, 1999). This mathematics education course tried to provide opportunities for preservice teachers to develop knowledge for teaching early algebra that is specific to their needs (Welder & Simonsen, 2011: 4).

b) Creating a learning community

Borko et al. (2005b) discuss the importance of participation in learning communities as a crucial element in meaningful learning. They identify three features of classroom life that promote successful learning communities: safe environment; rich tasks; student explanations and shared processes of learning. They suggest using initial activities to create discourse norms and to establish trust, followed by the long

term goal to establish a safe environment which helps create a culture that supports the sharing of ideas.

The first activity of the Maths 2 course started with a *Breaking the Ice* (getting to know one another) activity. Although the preservice teachers (PST) were all in their third year, they had not worked together before in a mathematics education classroom. They register for a Foundation or Intermediate/Senior Phase education degree and take separate mathematics education classes for the first two years. Previous experience of working with third year PSTs had alerted me to the need to actively engage students in getting to know one another and in developing a safe environment for mathematics learning.

The format of the lectures varied according to the topic and involved some or all of the following:

- Short demonstration lectures followed by written tasks and whole class discussion
- Interactive lectures which involved brainstorming ideas and encouraged PSTs to actively engage with the content
- Problem solving: lecturer posing questions or demonstrating a concept and posing “what if” questions

Preservice teachers were exposed to a variety of teaching strategies throughout the course of the year. Teaching was approached in different ways to cater for preferred learning styles, while at the same time demonstrating different approaches to the teaching of mathematics. The class of 26 students was divided into mixed phase groups of four/five and encouraged to work individually and collaboratively in class and to interact and support one another in and outside the classroom. There was opportunity for verbal and written reflection to assist in the construction of knowledge for teaching early algebra. They were given homework tasks after each lecture to reinforce new concepts, to perfect mastery, and to provide additional opportunity to engage with challenging problems without time restrictions.

c) Programme structure

Each year of the BEd degree is comprised of a 32 academic week year which is shared between input on academic content (24 weeks) and teaching practicum (8 weeks).

This is an institutional structure and looks as follow:

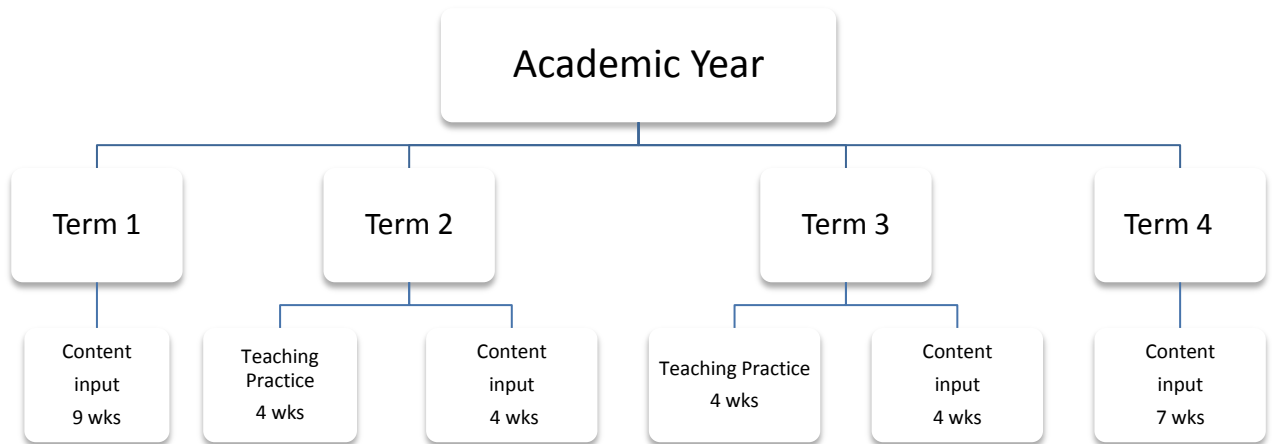


Figure 3.3: Institutional structure

The academic input is spread throughout the year and the teaching practicum which offers the preservice teachers the opportunity to connect theory to practice, is separated into two four week blocks. The Maths 2 course was organised such that during the first 9 weeks of the year, the PSTs were exposed to common content knowledge (CCK) as well as specialised content knowledge (SCK) for teaching early algebra after which they had to complete a four week teaching practicum in the schools. During the teaching practicum periods preservice teachers have to teach a number of lessons from different subjects, including mathematics, as prescribed by the tutor teacher. The content is usually specified by the host school and varies depending on the context and time of the year. I requested the PSTs, with permission from the WCED, the schools, parents and learners, to plan and teach an early algebra lesson. The purpose of teaching the lesson was to integrate content and theory from the Maths 2 course with teaching practicum to demonstrate the development of knowledge for teaching early algebra.

d) Course content and activities

Much of the focus of the early work in the Maths 2 course was to build on prior knowledge, develop connected, conceptual and procedural understanding as well as to build resourceful, self-regulating problem solvers (Kalchman & Koedinger, 2005). This involved four types of activities: solving mathematics problems, examining children’s thinking, reading and discussing current literature, and reflecting on one’s own learning. The mathematical problems focussed on key algebraic concepts

including a variety of open-ended and closed problems. They were often situated in real-life contexts and selected to allow for multiple solution strategies.

While the Maths 2 course had a strong CCK focus, the design of the course included opportunity to develop SCK through class discussions of mathematical content and pedagogy as well as through focused group discussions. The group discussions occurred once a week during lecture time for 15-20 minutes on six different occasions throughout the year. The preservice teachers were divided into groups of four/five and given articles linked to early algebra (EA) to read and prepare for subsequent lectures. The articles were taken from different sources and addressed a variety of aspects of EA which are important for teaching and learning.

The format of the group discussions was for a pre-selected group, who had prepared and studied the article intensely, to come and sit in the centre of the lecture room. The remainder of the class listened to the discussion but did not participate. I sometimes led the discussion with specific questions while on other occasions I encouraged the preservice teachers to take the lead. It was expected that the article was read and prepared for the discussion and each member of the group had to take turns to participate. The purpose of the discussion was to unpack and discuss key issues in the article and to highlight its relevance to early algebra. Once the 15 minutes was over, the remaining preservice teachers were encouraged to participate and ask questions or make comments.

A selection of six articles was discussed throughout the course of the year. These included three chapters from the Blanton (2008) book "Algebra and the elementary grades: Transforming thinking, transforming practice". The first chapter looked at what algebraic thinking is and what it is not. The second and third chapters focused on building early algebra skills by generalising arithmetic and functional thinking. The chapters included case studies of teachers from Foundation and Intermediate Phase; examples of activities that could be used in the classroom; and instructional strategies for making connections between arithmetic and algebra. We also discussed the Blanton and Kaput (2003) article which looked at developing children's capacity for early algebra thinking, including instructional tasks and application in the classroom. The Blanton and Kaput article was included to develop preservice teacher knowledge for teaching early algebra because it had a strong theoretical base and included examples of algebraic tasks and instructional

strategies that could be used to develop early algebra thinking. It highlighted alternative solutions to problems, provided examples of different representations and gave examples of how early algebra thinking could be planned for and developed in the primary school classroom.

The final two articles on early algebra were discussed later in the year and came from Stacey (2008) and Steckroth (2010). The former article examined aspects of the change from arithmetic thinking to algebraic thinking. It included several examples of children's mathematical thinking which illustrated the transition from arithmetic to algebra as well as examples of how arithmetic teaching could be transformed to better prepare children for algebra. The latter article showed how an elementary task for PSTs linked elementary school multiplication to the algebraic thinking of secondary school mathematics. Steckroth (2010) took the multiplication grid and showed how it could be linked to patterns, tables, inputs and outputs, area, formulae, graphs and calculus. It helped preservice teachers to see that "elementary school mathematics is not elementary at all; it is the cornerstone for all mathematics that students study in the middle grades and high school – even calculus" (p. 299).

The group discussions were used as a strategy to encourage discussion and reflection of the research articles but also as a strategy to develop knowledge for teaching early algebra. The preservice teachers kept a reflective journal for the duration of the Maths 2 course to capture their emerging thoughts and understanding of common content knowledge and specialised content knowledge for teaching early algebra. They had to write a brief summary of the main points that emerged in the discussion as well as their reflections after each of the group discussion sessions. The journals were collected three times during the year for informal assessment purposes and used for further feedback and discussion with the preservice teachers.

3.5.4. Data collection methods

This is an instrumental case study used to investigate the notion of knowledge for teaching early algebra in the context of preservice teacher education. The study is also descriptive in that it tries to describe and understand how knowledge for teaching develops as the result of the Maths 2 course and teaching practicum from the theoretical perspectives of knowledge for teaching and early algebra. Case

study research employs a variety of data collection methods which are selected to give a detailed and intensive description of the case. The choice of data collection methods is usually guided by the purpose of the research and the research questions. The most commonly used methods in case study research are interviews, observations and document analysis. Simons (2009: 43) outlines the aims and purposes of each method in some detail. She suggests that interviews are useful in helping to get to core issues more quickly and in depth, to probe and to ask follow-up questions which assist individuals to tell their stories. Observations help give a more comprehensive picture of the site, provide rich descriptions and help reveal the norms and standards of the institution or program. Videos have been used for some time to record classroom observations which allow for analysis after the event. Document analysis is not restricted to formal policy documents and public reports but can include anything written or produced from the context or site. Written documents can provide clues to understand the beliefs, values and attitudes of the writer.

Because I wanted to understand how knowledge for teaching early algebra develops, I used opportunities within the Maths 2 course to gather data from the preservice teachers. I did not conduct a pre-test to test their knowledge for teaching early algebra prior to the course as I did not want to conduct an evaluative study which would track their development over time. Rather I was interested in understanding how knowledge for teaching developed as the preservice teachers engaged with the content, theory and practice of the Maths 2 course. There were four types of data collected: video lesson recordings, lesson reflections, video questionnaires and focus group interviews.

Having had nine weeks of input in CCK and SCK for teaching early algebra, the preservice teachers prepared and taught an early algebra lesson to a specific grade during their teaching practicum. This was immediately followed by a post-lesson unstructured interview to discuss the lesson which formed the basis of the content used by the preservice teachers to write up their lesson reflections. Preservice teachers were later given copies of their video and a questionnaire to complete, and focus group interviews were conducted at the end of the course. The reasons for using different data collection methods at different stages of the year was to improve the quality measure of the study by looking for additional facets of the development of knowledge for teaching early algebra (Rule & John, 2011).

a) Video lesson recordings

The early algebra lessons were planned and designed by the preservice teachers informed by content, theory and activities from the Maths 2 course (see Appendix B: lesson plan). Preservice teachers also made use of curriculum guidelines, school textbooks and internet websites of early algebra lessons. The planned lessons were discussed during lectures and further checked by the tutor teacher for age appropriateness and academic demand. Access to schools was negotiated by me with the WCED, school principals, tutor teachers and parents, and permission was granted in writing for the videoing of the lessons during teaching time for research purposes (see Appendix C: letter WCED and schools). The early algebra lesson was videoed by a specialist camera technician while I took field notes of the lesson. The lessons were transcribed and transcriptions were checked with the videos to present accurate observations of the lessons and were used for later analysis.

b) Lesson reflections

Immediately after completion of the videoed lesson, I conducted one-on-one unstructured interviews with each of the preservice teachers for 20-30 minutes, to reflect on the overall lesson objectives and outcomes of the lesson. This included strengths and weaknesses of the lesson. I made summary notes as we discussed the lesson and gave each preservice teacher a copy of my notes. The preservice teachers then had 24 hours to type up their own reflection of the lesson which they sent to me electronically. The purpose of the task was to encourage preservice teachers to write out the mathematical purpose of the lesson (including content and pedagogy) and, through the discussion and reflection process, to further develop their knowledge for teaching early algebra. The lesson reflections provided written documentation of our discussion which could be analysed later.

c) Video questionnaires

The preservice teachers were given copies of their video lesson recordings at week 17 of the course. This was six weeks after the teaching practicum. It was time-consuming to process twenty-six videos and to make copies for each preservice teacher. The Maths 2 lectures continued in the interim and we had a number of interesting and lengthy discussions about the teaching practicum experience and the early algebra lessons. The lectures continued to address common content knowledge and specialised content knowledge for teaching early algebra. Once the video recordings were complete, I prepared a video questionnaire for the preservice

teachers with eleven questions focusing on specialised content knowledge for teaching early algebra. The rationale for the questionnaire was to help develop their knowledge for teaching early algebra by using the videos to encourage preservice teachers to reflect on the teaching episode drawing from the content, theory and discussion in class. I wanted them to consider different aspects of their lesson and made use of the MKfT model to guide their inquiry. Questions focused on learner answers, teacher explanations, learner solutions, language, learner contributions and difficulties and included the preservice teachers' design and plan of the lesson, choice of representations, as well as selection and sequencing of content. The final question asked preservice teachers to evaluate their lesson from their perspective and to justify their answer. I was interested in the preservice teachers' ability to self-assess given the lesson feedback discussion, the video footage and the video questionnaire. I was particularly interested what the questionnaires could tell me about the development of their knowledge for teaching early algebra.

d) Focus group interviews

The final data collection method took the form of focus group interviews. The purpose of the interviews was to capture the preservice teachers' overall thoughts of the Maths 2 course and teaching practicum and to link this to the development of knowledge for teaching early algebra. Simons (2009) identifies a number of advantages of focus group interviews: they are less threatening for individuals, economical for time, enables the researcher to gauge levels of agreement amongst the group, and provide a cross check of perspectives and statements of individuals. There are also some disadvantages which can emerge in the form of dominant individuals who take over the interview and are difficult to manage. It is not always easy to let all people have a say and it is necessary to watch out for "group think" (p. 49). However, it also offers insightful perspectives and observations on learning and teaching while encouraging reflection and discussion.

I interviewed two groups of preservice teachers on completion of the Maths 2 course and elected to interview the Foundation Phase and Intermediate/Senior Phase preservice teachers separately. I wanted to determine what they had learned from their phase perspective as the teaching of early algebra is different in each of the phases. They had worked in mixed phase groups in the Maths 2 course and had learned to interact through a community of practice nurtured in the lectures. The focus group interview was a volunteer process involving 9 FP students and 10 ISP

students from a class of 26. The interviews were semi-structured and each lasted 45 to 50 minutes.

The focus group interviews had two questions, the first of which dealt with their experience of the Maths 2 course in four topic areas: content, theory, methodology and teaching practicum. The second question looked at the legacy of the course and what in their view would remain with them in the long term. The responses were to be analysed later to illustrate preservice teachers' development of knowledge for teaching early algebra.

I prepared cards with the four topic areas as headings and placed them on a table for the whole group to see and make reference to later. The meaning of each of the topic areas was clarified and related to the context of the Maths 2 course and teaching practicum:

- *Content* referred to mathematical course notes, tasks, and assessments
- *Theory* referred to academic text taken from articles and books on early algebra
- *Methodology* referred to teaching strategies and classroom management
- *Teaching practicum* referred to their teaching of early algebra in the school

The preservice teachers were then invited to comment on each of the topic areas mentioned above. I explained their feedback would be used to inform my research and to improve the overall design and delivery of future Maths 2 courses. The preservice teachers participated enthusiastically and needed very little individual encouragement to participate. We moved to the next topic area once a topic area had been exhausted. The interviews were audio recorded and transcribed for analysis later.

3.5.5. Data analysis

Yin (2009) identifies five techniques for data analysis in qualitative research: pattern matching, explanation building, time-series analysis, logic models, and cross-case synthesis. Pattern matching involves comparing patterns within the data and with patterns in the theory (or alternative predictions). If the patterns concur, the results help strengthen the internal validity of the case study. Explanation building is a special type of pattern matching in which the goal of the analysis is to build an explanation about the case. The next two techniques are used in experiments and

evaluations and are not relevant to this study and the fifth technique applies specifically to the analysis of multiple cases which is not appropriate here. I decided to use pattern matching as it gave me the opportunity to look for patterns within the data and to match these with my theoretical framework: knowledge for teaching early algebra.

The data set included four sources of data: 26 video recordings of early algebra lessons, 26 lesson reflections, 26 video questionnaires and 2 focus group interviews. This data was collected at different stages of the academic year with the specific purpose of looking at the development of knowledge for teaching early algebra using a variety of data collection methods. The analysis of the data in this study focused on providing a description to understand the development of preservice teachers' knowledge for teaching early algebra. This involved looking for patterns within the different sources of data to locate development of knowledge for teaching early algebra through the preservice teachers' espoused and enacted responses. Using different methods of data collection throughout the year helped show up additional facets of development of knowledge for teaching early algebra (crystalisation). Although the focus of the study was understanding the development of the preservice teachers' knowledge for teaching early algebra, I was initially reluctant to code until I had an overall sense of what was emerging from the data. I started with the lesson reflections as they were written accounts of the preservice teachers' sense of what had happened in the early algebra lessons they presented and issues that were pertinent for them.

a) Lesson reflections

The lesson reflections were completed by the preservice teachers within twenty four hours of the taught lesson. They were a summary of the post-lesson interview between me and the preservice teacher and included personal reflections on the strengths, limitations and possible improvements to the lesson.

Level 1 analysis

The analysis started with my reading of the lesson reflections to get a general overview of the full data set of the 26 preservice teachers and to locate issues raised in the lesson reflections. Much of the initial patterning seemed to highlight issues of content and pedagogy with comments and suggestions for future improvements to the lesson. Using a recursive approach to the analysis, I decided

to go back to my research question and theoretical framework and elected to code the data using two categories: *early algebra* (common content knowledge (CCK)) and *mathematical knowledge for teaching* (specialised knowledge for teaching (SCK); Horizon knowledge (HK); knowledge of content and students (KCS); knowledge of content and teaching (KCT); knowledge of content and curriculum (KCC)).

The early algebra codes and descriptions were taken from the work of Blanton et al. (2005a) in algebraic thinking and a selection of the codes were utilised for their applicability to this study. The knowledge for teaching codes and descriptions were taken from Ball, Bass & Hill (2004) as well as Ball et al. (2008) and a summary of the codes and descriptions can be found in Table 3.1 below. I used Atlas.ti, a qualitative computer research package, to manage the coding of the data and to generate some frequency measures. The measures were not used for statistical purposes but rather to help give an overview of the nature of preservice teachers' responses.

The first level of analysis produced 33 references to early algebra and 73 references to knowledge for teaching. I did not find any references to knowledge of content of the curriculum (KCC) or horizon knowledge (HR) and did not include codes for these in Table 3.1.

Table 3.1: Summary code list – Lesson Reflections 1

CODE	DESCRIPTION	FREQUENCY OF COMMENTS
Early Algebra		33
AT- CAT B: Properties	Exploring properties of operation on whole numbers e.g. $a+b=b+a$	3
AT- CAT C: Equality	Exploring equality as expressing a relationship between quantities	2
AT- CAT E: Equations	Solving missing number sentences	1
AT- CAT F: Symbol	Symbolising quantities and operating with symbolised expressions e.g. $x + y = 8$ or $x+5=9$	2
AT- CAT G: Graph	Representing data graphically	1
AT- CAT H: Rule	Finding functional relationships	13
AT- CAT I: Predict	Predicting unknown states using known data in other words making conjectures about unknown states e.g. what happens when you have a hundred people	7
AT- CAT J: Identities	Identifying and describing numerical and geometric patterns	4
Mathematical knowledge for teaching		73
KCS: Knowledge of content and students	Anticipating student thinking and confusions Selecting tasks that they will find interesting Anticipating what they will do with a task Predicting if they will find a task easy or difficult Recognising errors and misconceptions interpret emerging and incomplete thinking	8
KCT: Knowledge of content and teaching	Sequencing and selection of tasks Choosing effective instructional models Knowing the advantages of different representations Using different methods and procedure to make maths more salient and usable Managing discussion and feedback Pursue, ignore or save student contributions Know what language and metaphors can assist and confuse students	21

CODE	DESCRIPTION	FREQUENCY OF COMMENTS
SCK: Specialised content knowledge for teaching	Presenting mathematical ideas Responding to 'why' questions Finding appropriate mathematics examples Representations(R): recognising what is involved in using a particular R, linking R, selecting R for particular purposes, choose, make and use mathematics R effectively Connecting topics to prior or future topics Appraising and adapting textbooks Modifying tasks to make easier or more difficult Evaluating student responses Giving or evaluating mathematics explanations Choosing and developing definitions Using and critiquing mathematics notation and language Explaining how mathematics language is used e.g. edge in maths Asking productive mathematics questions Understanding different student interpretations of problems and solutions Making content visible to and learnable by students Explain and justify mathematical ideas Know and understand alternative algorithms Analyse common errors and misconceptions from mathematical perspective	44

Level 2 analysis

The level 1 analysis helped to code the data from the lesson reflections but did not appear to capture the richness of the data. The early codes were too restrictive and did not cover the full range of preservice teachers' conceptions of knowledge for teaching early algebra as indicated in the lesson reflections. This required a re-read of the theory and literature related to the knowledge for teaching model and early algebra which helped formulate a more comprehensive set of codes and descriptions focusing specifically on CCK and SCK of early algebra and was in line with the research purpose and research questions.

Using the early algebra literature from a variety of sources (Kieran, 1996; Blanton & Kaput, 2005a; Carraher & Schliemann, 2007; Mason, 2008; Smith, 2008) I formulated a list of descriptions, which I called CCK, to reflect the diversity of

understandings of early algebra and used this to help analyse the lesson reflections of the preservice teachers. I also reconsidered the descriptions for SCK using the six tasks of teaching as outlined by Kazima et al. (2008). I then re-read the Ball et al. (2004, 2005, 2008) articles to compile a comprehensive list of descriptions for each of the teaching tasks as reflected in Table 3.2. I decided to exclude the KCS and KCT codes, which although interesting, were not relevant to this study. The process of redefining the categories required a dialogical approach between the theory and the research which meant that theory infused all aspects of the analysis. It helped to inform the analysis while at the same time it highlighted the ambiguous and complicated nature of describing and understanding knowledge for teaching early algebra.

I used Atlas.ti to analyse the lesson reflections (see Appendix D for coding examples) a second time for CCK and SCK of early algebra using the descriptions summarised in Table 3.2.

Table 3.2: Summary list – Lesson Reflections 2

CODE	DESCRIPTION	FREQUENCY OF COMMENTS
CCK		54
Early Algebra	Generalised arithmetic Study of functions, relations and joint variation Reasoning about change Modeling of real life situations Quantitative reasoning Abstraction and generalisation Representational thinking Working with relationships Global meta-level activity: analysing, noticing structure, studying change, problem solving, justifying, proving and predicting Making connections	
SCK		147
Definitions	Using mathematically appropriate and comprehensible definitions	0

Explanations	Using mathematical notation and language and critiquing its use Connecting topics to prior and future learning Design mathematically accurate explanations that are comprehensible and useful for learners Giving and evaluating maths explanations It involves making features visible to and learnable by learners e.g. place value	23
Questions	Posing good mathematical questions and problems that are productive for learners' learning	24
Representations	Represent ideas carefully: Physical, graph, symbol, and operation Making connections among representations Selecting representations for particular purpose Make and use mathematical representations effectively	43
Tasks	Modifying tasks to make easier or more difficult Make judgments about the mathematical quality of instructional materials and modify as necessary. Finding appropriate mathematical examples	27
Working with learner ideas	Interpret and make mathematical and pedagogical judgements about learners' questions, solutions, problems and insights Responding productively to learners' questions Assess learners' mathematical learning and take the next steps	30

A sample of lesson reflections were coded and verified for SCK by two independent mathematics educators using the codes and descriptions given above. The CCK descriptors were not included as it would have required the evaluators to have a detailed and thorough knowledge of the early algebra literature which was not possible in the given time frame.

b) Video questionnaires

The preservice teachers were given a video copy of their early algebra lesson, six weeks after the original event, with a set of questions to be answered by the preservice teachers and to focus the analysis of their lesson. The questions were used to understand the preservice teachers' development of knowledge for teaching early algebra over time, in terms of early algebra and specialised knowledge for teaching. The questions focused primarily on specialised knowledge for teaching early algebra as it is the knowledge that is used in classroom settings and is needed by teachers in order to teach effectively (Ball et al., 2004). This is not to ignore the need for common content knowledge needed for teaching early algebra but to get a

better understanding of the specialised knowledge for teaching early algebra. The questions were as follows:

1. What Early Algebra knowledge and skills did you draw on to plan for and use in this lesson?
2. How did you make features of particular content visible to and learnable by learners?
3. Identify ONE episode in the video when learners gave different answers (correct/incorrect). Describe how you handled this situation and explain why you handled it the way you did.
4. How did you help learners to explain and justify their mathematical ideas?
5. How did you help learners to make sense of other learners' solutions?
6. Can you identify episodes in the video when you tried to develop learners' algebraic language? Give a brief description of one such episode.
7. Describe the mathematical representations that you chose and explain why you made this selection.
8. What influenced your selection and sequencing of activities?
9. How did you decide what learner contributions to pursue and which to ignore or save for later?
10. What learner difficulties and/or errors did you anticipate? How did you provide for this in the planning of the lesson?

The responses were coded (see Appendix D for coding examples) using the codes and descriptions discussed previously and results were given as follows. Again the frequency measures were not used to measure the CCK and SCK of preservice teachers but to reveal the nature of their responses:

Table 3.3: Summary list of codes for video questionnaires

CODE	FREQUENCY OF COMMENTS
CCK	13
Early Algebra	13
SCK	161
Definitions	0
Explanations	17
Questions	0
Representations	39
Tasks	17
Working with learner ideas	88

The questionnaire contained only one question related to early algebra content, language, tasks and representations and five questions related to working with learner ideas, hence the seemingly skewed nature of the data.

c) Video recordings

The videos were analysed at the end of the analysis process to understand the development of knowledge for teaching early algebra through the actions of the preservice teachers in the classroom. The planned and taught early algebra lessons gave the preservice teachers an opportunity to engage with the practice of teaching early algebra and to apply their common content knowledge and specialised content knowledge in the mathematics classroom. It also provided an opportunity for preservice teachers to later reflect on the lesson and to answer questions related to their specialised content knowledge for teaching in the hope of developing knowledge for teaching early algebra.

The analysis of the videos provided an opportunity to engage with their enacted knowledge of teaching early algebra as distinct from their espoused knowledge as indicated in the lesson reflections, video questionnaires and focus group interviews. The purpose of the video analysis was to investigate practice, the actual work of teaching, to identify tasks that were prevalent in the lesson and highlight the mathematical demands and opportunities for teaching early algebra these tasks presented for the preservice teachers. Thames et al. (2008: 3) reminds us that

elementary teaching is “highly mathematical work” requiring “substantial mathematical knowledge and reasoning”.

Five video recorded lessons were identified through systematic sampling to represent the continuum of teaching early algebra from Grade 3 to Grade 7. The videos were selected based on the variety of teaching tasks in the lesson as well as the demands and opportunities that arose during teaching. The selected examples were not evaluative of the preservice teacher but helped elucidate the developing mathematical knowledge for teaching early algebra as demonstrated by the preservice teachers.

Once the relevant teaching tasks, such as giving explanations, using representations, questioning, restructuring tasks and working with learners’ ideas, were identified these were further analysed for knowledge for teaching early algebra (Kazima et al., 2008). The Smith (2008) framework for functional thinking and the Blanton & Kaput categories of functional thinking were used to analyse the development of algebraic thinking in the pattern and functions lessons. They helped give a structure to the analysis which investigated the preservice teachers’ specialised knowledge for teaching functions as it was used in the classroom setting and needed by the teacher in order to teach effectively (Ball et al, 2008).

d) Focus group interviews

The focus group interviews were analysed for common content knowledge, specialised content knowledge for teaching early algebra, theory (literature in the Maths 2 course), and teaching practicum (see Appendix D for coding examples). The theory and teaching practicum categories were included to understand which elements of the Maths 2 course and teaching practicum could be identified as influential in the development of mathematical knowledge for teaching, as perceived by the preservice teachers.

The results were as follows and again are not to be used for statistical analysis but rather to indicate the general nature of the responses.

Table 3.4: Summary list of codes for focus group interviews

CODE	FREQUENCY OF COMMENTS
CCK	13
Early Algebra	13
SCK	30
Definitions	0
Explanations	9
Questions	4
Representations	5
Tasks	5
Working with learner ideas	7
Theory	12
Teaching practicum	30

e) Summary of analysis

The overall research question was to understand the development of preservice teacher knowledge for teaching early algebra as a result of the Maths 2 course and teaching practicum. This involved an analysis of preservice teachers' verbal and written responses (lesson reflections, video questionnaires and focus group interviews) and their actions in the classroom (video lesson recordings) for manifestations of knowledge for teaching early algebra and was guided by the following research sub-questions:

Research sub-questions:

1. What manifestations of knowledge for teaching early algebra do preservice teachers illustrate in their verbal and written feedback taken from an early algebra course and teaching practicum?
2. What manifestations of knowledge for teaching early algebra do preservice teachers demonstrate in their teaching of an early algebra lesson during teaching practicum?
3. What aspects of the early algebra course and teaching practicum contribute to the development of knowledge for teaching early algebra?

The analysis process was threefold: it started by looking across the verbal and written responses (reflections, questionnaires and interviews) of the preservice teachers for manifestations of their knowledge for teaching early algebra. The responses were then grouped into common content knowledge and specialised content knowledge of early algebra guided by the theoretical framework of the study. These were then analysed further to understand the development of knowledge for teaching.

Secondly, the analysis process involved looking for manifestations of knowledge for teaching early algebra as demonstrated in case studies of five preservice teachers' video recordings of early algebra lessons and focused exclusively on specialised knowledge for teaching early algebra. This process is described in more detail in Chapter 4.

The final stage of the analysis dealt with the focus group interviews and involved looking for manifestations of CCK and SCK, as well as identifying aspects of the Maths 2 course and teaching practicum that were helpful in the development of knowledge for teaching early algebra as perceived by the preservice teachers.

The findings from the analysis of the data were then organised and discussed in four categories:

- Common content knowledge of early algebra
- Specialised content knowledge of early algebra
- Specialised content knowledge in teaching an early algebra lesson
- The Maths 2 course

The first two categories deal with the preservice teachers' espoused knowledge for teaching early algebra (CCK and SCK) as illustrated in the verbal and written responses taken from the reflections, questionnaires and interviews. This is followed by case studies of preservice teachers' early algebra video lessons which demonstrate different aspects of knowledge for teaching early algebra enacted in the classroom. The manifestations were used to understand the development of knowledge for teaching early algebra during and after the Maths 2 course and teaching practicum. The preservice teachers also discussed possible reasons for

the development of knowledge for teaching early algebra during the focus group interviews. These are discussed further in Chapter 4.

3.6. Conclusion

This chapter began with a review of the research question and purpose of the study and included an explanation of the research design. The research was qualitative within an interpretive paradigm which impacted on the choice of research methodology. A case study methodology was used to investigate the development of knowledge for teaching followed by an explanation of the research methods employed in the study, including descriptions of the site, sample and the Maths 2 course. Data collection and analysis involved the use of interviews, observations and document analysis which were explained and justified in the context of the study. The description of the data analysis concluded with an overview of the research questions and categories of analysis. The findings of the research will be unpacked and discussed in more detail in the next chapter.

CHAPTER 4. FINDINGS

4.1. Introduction

This chapter contains a summary of the findings and interpretation of the development of preservice teachers' knowledge for teaching early algebra. Knowledge for teaching is defined as the common content knowledge and the specialised content knowledge needed to teach early algebra. The chapter starts with a summary of the mathematical knowledge for teaching (MKfT) model to give an overview of key elements of the theory and its link to this study (4.2). This is followed by an explanation and demonstration of the manifestations of knowledge for teaching early algebra, both CCK and SCK, as illustrated by the preservice teachers' verbal and written responses (4.3 and 4.4). These were taken from the preservice teachers' lesson reflections, video questionnaires, and focus group interviews. The next section (4.5) of the chapter deals with the findings from the preservice teachers' video recorded early algebra lesson and looks at the specialised knowledge for teaching which emerges from the lesson. This is presented in the form of case studies and includes illustrations taken from the early algebra lessons. The final section (4.6) deals with findings from the preservice teachers' focus group interviews about the Maths 2 course and concludes with a summary of the chapter.

4.2. Summary of mathematical knowledge for teaching model (MKfT)

As mentioned in Chapter 2, the more recent work of Ball and her colleagues (2008) is an elaboration of Shulman's notion of PCK within the context of mathematics education and seeks to answer questions regarding the composition and structure of mathematical knowledge for teaching (MKfT). The results of a number of studies suggest that teachers' knowledge of content is closely related to practice and the work that teachers do (Shulman, 1986; Ma, 1999; Ball et al., 2008). Thames et al. (2008:2) identify three major lines of research that have contributed to the development and understanding of Ball and her colleagues practice-based theory of MKfT:

1. Analysing classroom teaching to see what mathematical knowledge arises in the work teachers do;
2. Developing theoretical domains and instrumentation for detecting, measuring and categorising mathematical knowledge for teaching

3. Teaching MKfT to teachers and other professionals who work with teachers in various settings.

Their research has provided evidence that there is mathematical knowledge specific to teaching and that it can be measured. Furthermore, their findings show that a teacher’s mathematical knowledge for teaching is linked to the quality of instruction and is a significant predictor to gains in student achievement (Hill, Rowan & Ball, 2005).

MKfT is defined as the mathematical knowledge needed to “perform the recurrent tasks of teaching mathematics to students” i.e. the mathematical knowledge “entailed by teaching” (Ball et al., 2008: 97). The term “need” is expanded to include the perspective, habits of mind, and sensibilities that matter for effective teaching of mathematics. Mathematical knowledge for teaching is more expansive than pedagogical content knowledge in that it includes both common content knowledge and specialised content knowledge of the subject taught (Hill et al., 2012). It does not seek to replace Shulman’s notion of PCK but rather to elaborate on the fundamentals of subject matter knowledge for teaching (Thames et al., 2008).

The MKfT model (Ball et al., 2008) distinguishes between subject matter knowledge (SMK) and pedagogical content knowledge (PCK) and defines six domains of knowledge for teaching: common content knowledge (CCK); horizon knowledge; specialised content knowledge (SCK); knowledge of content and teaching (KCT), knowledge of content and students (KCS) and knowledge of content and the curriculum.

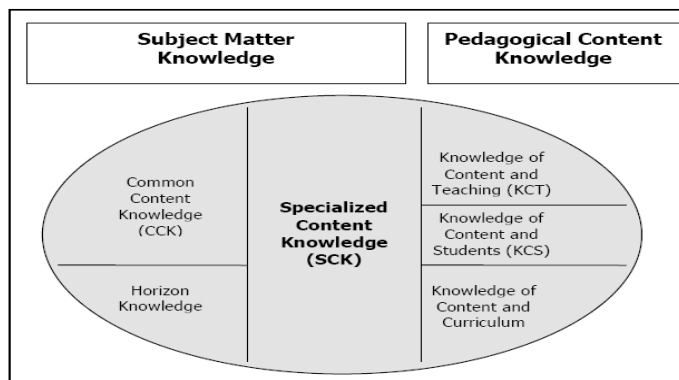


Figure 4.1: Domains of mathematical knowledge for teaching

The purpose of this study was to understand the development of preservice teachers' knowledge for teaching early algebra as a result of an early algebra course and teaching practicum using the MKfT model. The main research question and sub-questions which guided this study were as follows:

Research question:

How does preservice teachers' knowledge for teaching early algebra develop as a result of an early algebra course and teaching practicum?

Research sub-questions:

1. What manifestations of knowledge for teaching early algebra do preservice teachers illustrate in their verbal and written feedback taken from an early algebra course and teaching practicum?
2. What manifestations of knowledge for teaching early algebra do preservice teachers demonstrate in their teaching of an early algebra lesson during teaching practicum?
3. What aspects of the early algebra course contributed to the development of knowledge for teaching early algebra?

The development of knowledge for teaching was monitored through the use of video recordings of early algebra lessons, lesson reflections, video questionnaires and focus group interviews during a year long course. Each set of data was analysed for common content knowledge (CCK) and specialised content knowledge (SCK) using the verbal and written responses and actions of the preservice teachers. The descriptors for the CCK and SCK codes were drawn from the theoretical framework of the study (both early algebra and the MKfT model). The CCK and SCK findings were then clustered into common themes which emerged from the data and linked back to the theory on early algebra and mathematical knowledge for teaching to form a dialogical approach between theory and research (Rule & John, 2011). The findings from the analysis of the data were then organised into four categories:

- Common content knowledge of early algebra
- Specialised content knowledge of early algebra
- Specialised knowledge in teaching an early algebra lesson - Case study examples
- The Maths 2 course

The first two categories deal with the preservice teachers' *espoused* common content knowledge (CCK) and specialised content knowledge (SCK) of early algebra as illustrated in the verbal and written responses taken from the lesson reflections, questionnaires and interviews. This is followed by selected video lesson examples of early algebra teaching which demonstrate different aspects of mathematical knowledge for teaching *enacted* in the classroom. The final category looks at aspects of the Maths 2 course which contributed to the development of knowledge as indicated from the preservice teachers' responses.

4.3. Preservice teachers' common content knowledge (CCK) of early algebra

Ball et al. (2008: 399) define common content knowledge (CCK) as the "mathematical knowledge and skill used in settings other than teaching". Teachers must know the mathematics they teach, be able to use correct definitions, mathematical language and notation, and recognise mathematical errors. CCK is not specialist knowledge that only teachers possess, but mathematical knowledge that can be used by other people, in other settings. The use of the word "common" does not imply that all people have this knowledge but it is knowledge that is not unique to teaching and can be held by others that know mathematics.

The use of CCK in this study refers to the knowledge of early algebra as a "rich sub-domain of mathematics education" (Carraher & Schliemann, 2007: 673). While there is common agreement on what algebra children need to know by the end of high school, there is "less agreement" in the case of early algebra (p.673). Early algebra research has grown over the past twenty years and arose from the difficulties that high school learners were experiencing with algebra (Booth, 1988). Those who support the EA approach feel that the problems in learning algebra are less about the algebra and more about the lack of connectivity between arithmetic and algebra in the younger grades (Carpenter et al., 2003; Blanton & Kaput, 2005a, 2008; Carraher et al., 2006, 2008b).

There is a plethora of explanations of what EA is. This has been elaborated upon in some detail in Chapter 2. Early algebra or the development of algebraic thinking in the early grades is essentially one and the same concept. The analysis of preservice teachers' understanding of CCK of early algebra involved the coding of the verbal and written responses using Atlas.ti, a qualitative analysis software package. The selection of data used as illustrations of CCK – EA (common content

knowledge of early algebra) emerged from the verbal and written responses of the preservice teachers and were matched to current research in the field of early algebra.

The preservice teachers' illustrations of CCK of early algebra in the verbal and written responses could be linked to their prior knowledge of algebra as well as the content of the Maths 2 course. The Maths 2 course exposed preservice teachers to various perspectives of early algebra through investigation of pattern generalisation tasks as well as findings from empirical and theoretical research in the field. It was impossible to determine which set of experiences most influenced their common content knowledge of early algebra.

The work of Kaput (2008), a pioneer of EA research and well known theorist, suggests that algebra involves two reasoning process, the first using symbols to generalise and the second acting on symbols to generalise. These are then used in different ways depending on which approach to early algebra is being used (Kaput, 2008: 10)

- algebra as generalised arithmetic
- algebra as functional thinking
- algebra as modelling.

Similarly, Carraher & Schliemann (2007) identify three different entry points into EA: arithmetic and numerical reasoning; arithmetic and quantitative reasoning; and arithmetic and functions. The Maths 2 course focused on the last entry point: algebra as the study of functions, relations and joint variation to include using and acting on symbols to generalise. It involved "placing functions at centre of algebra instruction", "conceiving letters as variables" and "expressions as rules for functions" (p.687). This suggests that functions can provide a solid foundation for EA through the investigation of pattern activities with an emphasis on "formation of functional relationships and the generalisation of patterns" (Dooley, 2009: 441).

The next part of this section reflects a summary of the findings of the CCK of early algebra from the preservice teachers' verbal and written responses. The data was analysed for manifestations of common content knowledge (CCK) of early algebra. The manifestaions were then grouped into common categories and linked to the

definitions of early algebra available in the literature. Five categories of knowledge of EA emerged from the PST verbal and written responses. Each individual EA category begins with an explanation from literature of the relevant aspect of early algebra, followed by illustrations taken from the preservice teachers' verbal and written responses. An interpretation of the findings is included at the end of the section in the summary of the preservice teachers CCK of early algebra.

4.3.1. Early algebra as generalisation

Moss & Beatty (2006: 194) suggest that pattern activities offer a “powerful vehicle for understanding the dependent relations among quantities that underlie mathematical functions” as well as a “concrete and transparent way for young students to begin to grapple with the notions of abstraction and generalisations”.

The mathematical content of each of the preservice teachers' lessons focused almost exclusively on pattern generalisation activities. The illustrations given below indicate knowledge of EA and the importance of working with relationships, and moving towards abstraction and generalisation.

I needed to also place an emphasis on algebraic thinking and working with relationships. (S7 – VQ: Student 7, Video Questionnaire)

I started from the beginning getting the learners to realise that algebra is about relationships. (S6 – LR: Lesson Reflection)

The learners also saw patterns even in the symbolic form and they understood that it is not just a letter, but a representation of something. (S9 - LR)

.....generalising techniques were required from the students. (S12 - LR)

4.3.2. Early algebra as a process of functional thinking

Smith (2008: 143) describes functional thinking as “representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships to generalisation for that relationship across instances”. The purpose of a pattern activity is to specialise i.e. moving from the simple particular case to “see what is going on”, in order to generalise for a collection of particular cases (Mason, 2005:23). The development of functional thinking makes use of a “wide range of symbol systems including

tables, graphs, and various pedagogical systems such as ‘function machines’” (Kaput et al, 2008: 14).

The following are illustrations of preservice teachers’ knowledge of early algebra as functional thinking. It involves engaging with a functional situation, creating a record, seeking patterns and creating mathematical certainty.

The focus of my lesson was identifying the common difference and recognising the pattern, and recording information in a table through the matches activity (square) learners had to see, observe and do. Maybe if learners had to first create the 4th, 5th and 6th square with the matches on their own and see if they can identify and pick up a pattern (add 3) maybe this would have been better. (S10 - LR)

A few good things managed to come out of the lesson. Learners were able to identify the algebraic pattern. They were able to construct a function rule. Learners learnt how to use the spider diagram (version of the function machine) as well as the table method. For the homework task, they were able to solve the problem given to them using the table method which I was impressed with. They also managed to come up with a few brilliant algebraic problems that they had to construct themselves. (S2 - LR)

....my focus was to develop the skill of finding a formula using various pieces of information. (S3 - VQ)

I wanted learners to be exposed to a function/data table so that the patterns and relationships would be more visible. I wanted them to see the different types of relationships of the data by looking down and across at the data in order to understand how the values change in relation to one another. I wanted to expose learners to the knowledge and skills of making conjectures about the patterns and relationships they observed. They needed to see the power of it and how they could apply their assumptions to determine any value without knowing previous values. (S15 - VQ)

4.3.3. Early algebra as modelling

Sanchez & Llinares (2003:6) identify two concepts of functions: “function as action and function as model for a real situation”. The concept of function as a correspondence between sets is generally understood but there is additional emphasis on two different aspects of the mathematical concept for the purpose of

teaching. The first aspect is function as action which involves functions as a “chain of operations” and stresses the more formal aspects of algebra manipulation (p. 6). Secondly, there is function as model of real life situations which places emphasis on the “meaning of the relationship between the variables in real-world situation problems” (p. 14).

From the illustrations given below, there is data to suggest these preservice teachers felt it was important to consider the function concept as a real situation model and an important aspect of teaching. EA is about making algebra real and not abstract and putting mathematics into a context to improve accessibility for learners.

My goal upon approaching this school was to show them that algebra is real and not abstract and this is the direction that I followed in my Mathematics lesson - to put Mathematics into context. (S8 - VQ)

This lesson incorporated the knowledge and skills needed to recognise and name patterns as well as to extend it and recognise it in real life context. (S11 - VQ)

I found that once I had a real-life context to work from, everything seemed to work out and make a little more sense, especially for the learners when they made connections from the story and then linking up the context allowing them to focus on the algebraic aspect so much more. (S17 - LR)

A concept that worked really effectively was relating the math in the classroom to real-life examples/contexts. I made use of a concept that most of the learners could relate to, or otherwise, had heard of (piggy banks, pocket money, saving money). (S24 - LR)

4.3.4. Early algebra as global meta-level activity

Another of the categories to emerge from the data relates early algebra and global meta-level mathematical activity. Kieran (1996) categorises school algebra into three activities: generational activities, transformational activities and global meta-level activities. According to Kieran’s model (1996), generational activities of algebra involve the forming of expressions and equations that are the objects of mathematics, i.e. meaning-building for algebra. Transformational activity is concerned with changing the form of an expression or equation to maintain equivalence. Global meta-level mathematical activities are those in which algebra is

used as a tool. Kieran (2004: 149) offers a definition of algebraic thinking in the early grades which is based on her global meta-level activity of algebra and involves the development of ways of thinking such as “analysing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting”.

The illustrations below capture the responses of different preservice teachers and highlight the importance of moving from direct instruction to emphasise inquiry and higher levels of thinking allowing learners to conjecture, generalise and problem solve.

....exploring with them (learners) and allowing them to make their own conjectures, their own decisions, explore with them and not just let them sit back and say like that, that, that.(FGI - Focus Group Interviews)

I think the whole thing of the algebraic thinking and generalising process, rather than the product, will help with my teaching and my thinking when it comes to maths because I found it very interesting that generalising formulas and algebra is not just like okay here is the formula you do it. It's you coming up with way of doing it and how you do it and justifying, and justifying what you're talking about and setting it in a real world context. (FGI)

Yeah, I think there is a procedure in doing something but it's the thinking behind the procedure that we need to stimulate and I think we were more successful in doing it with this lesson (pattern lesson). (FGI)

4.3.5. Early algebra as a link between arithmetic and functions

The research findings of Moss and Beatty (2006) and Schliemann et al. (2008) show that pattern activities which have a focus on “transformation rules and numerical and geometrical representations may constitute a meaningful entry point to EA” (Carragher & Schliemann, 2007: 694). The concept of function can help to unite seemingly disconnected topics, for example using arithmetic operations as instances of functions eg. using the multiplication tables as embodiments of the function $a \times b$ and functions such as $7x + 3$ as an extension of the “times 7” table.

The illustrations given below demonstrate preservice teachers' knowledge of the link between arithmetic and algebraic thinking in which they think of “arithmetical

operations as functions rather than mere computations on particular numbers” (Carraher & Schliemann, 2007: 694).

I tried to make them more aware of different types of patterns and allowing them more practice with recognising the pattern in their numeracy work. (S13 - LR)

Most learners understood the concept of patterns and relationships and the idea of multiplication/ times tables being a possible solution. (S23 - LR)

I liked the fact that it was possible to link the concepts of doubling and halving to other mathematical concepts like patterns and algebraic thinking. (S15 - LR)

I decided to go back to some previous work done about three lessons ago where I put the 3 and 8 times table in a flow diagram, and learners were challenged to give the n^{th} term for both. They then realised that the 3 and 8 times table can be written as $3n$ and $8n$ respectively. This knowledge I used when getting them to find the pattern formula $y = 3n + 1$. Once they had completed the table and noticed the constant difference of +3, I ask them where else they find it? Their reply was the 3 times table which could be written $y = 3n$. I then got them to compare the output of $y = 3n$ to their pattern and they were easily able to see that the difference is +1 every time and they were then able to construct the formula $y = 3n + 1$. (S1 - LR)

4.3.6. Summary of preservice teachers' CCK of early algebra

Common content knowledge (CCK) of early algebra was organised into five categories based on verbal and written responses given by the preservice teachers. The five categories of knowledge represented an understanding of early algebra as the study of relationships (generalisation) and development of functional thinking. Early algebra also involved linking arithmetic thinking and algebraic thinking, using real life contexts and engaging in global meta-level activity.

There is no one vision or perspective of algebraic thinking in the early grades that has been adopted by the international community (Kieran, 2004). However the preservice teachers' responses touched on current understanding of EA as demonstrated in the illustrations and is linked to literature used in the Maths 2 course. The analysis of their illustrations indicated an understanding of a wide range of aspects of early algebra both related to current research in the field and their experience in the classroom. It highlighted the importance of understanding

patterns as the study of change, as the search for relationships and structure, and as the link to abstraction and generalisation. The use of pattern and function activities required learners to engage in problem solving, modelling and reasoning with an emphasis on conjecture, justification and explanation. There was a strong advocacy for linkage to real life contexts and to the use of representations to understand functions in its different ways i.e. real-life, table, verbal, symbol. However, there was very little linkage between arithmetic thinking and algebraic thinking and the focus remained very much on pattern and functions.

While the preservice teachers' illustrations appear to indicate development of CCK of early algebra, it is imperative to remember that these responses are held by individuals within a class and do not necessarily represent the same common content knowledge (CCK) for each and every preservice teacher. They exemplify the range of common content knowledge of early algebra held by this group of students.

The following set of extracts from the verbal and written responses presented a different aspect of common content knowledge and highlighted an important issue in the inter-relationship between CCK and SCK. The illustrations refer to episodes within the early algebra lessons and give an indication of the reflective ability (understanding) of preservice teachers which cannot be categorised specifically as CCK or SCK only. Although the responses seemed to deal primarily with pedagogical content issues, they appear to demonstrate a link between CCK and SCK.

The illustrations demonstrate knowledge of functional thinking, and the relevance of pattern activities to provide "opportunity for individuals to engage in functional thinking" (Smith, 2008: 145).

Generalising the formula was extremely challenging for them. To guide them I wrote the answers and asked them to find the relationship (what more do we need to get to our number?). (S3 - LR)

For the conclusion, I tried to get see if the learners were able to find the rule, but they were not able to, so I guided them to the right answer, which was not the absolute correct thing to do but they all did seem interested even though they were not thinking themselves. (S13 - LR)

Some learners had difficulty finding the formula or understanding the formula and what n stood for or what to do with it and many struggled to link the pattern with the formula. (S14 - LR)

When the learners were busy with their activity, many of them struggled with writing the functional rule in words. I should have paid more attention to that. Perhaps a spider diagram (function machine) would have helped in this regard as the functional rule is clearly stated on the diagram. (S23 - LR)

The extracts highlight the difficulty of working with pattern activities and show that preservice teachers are aware of the mathematical conceptual challenge of generalisation i.e. finding the function rule (Moss & Beatty, 2006). They also show knowledge of the mathematical activities underlying functional thinking which involve the focus on relationships between two (or more) varying quantities moving from “specific relationships to generalisations of that relationship across instances” (Smith, 2008: 143). As mentioned previously, it is difficult to categorise this knowledge as exclusively CCK or SCK but it is knowledge and awareness that is relevant to mathematics and mathematics teaching.

4.4. Preservice teachers’ specialised content knowledge (SCK) for teaching early algebra

The second category of content knowledge is specialised content knowledge (SCK): the mathematical knowledge and skill unique to teaching (Ball et al., 2008). It involves the unpacking of mathematics to make it accessible and understandable for learners and requires the teacher to have a “deep and connected understanding of mathematics and relationships among ideas” (Thames et al., 2008; Bair & Rich, 2011: 295).

I used the Kazima et al. (2008: 288) list of six tasks of teaching, which is based on the work of Ball et al. (2008), to analyse the verbal and written responses of the preservice teachers in looking for illustrations of specialised content knowledge for teaching early algebra. The six categories of tasks for teaching are as follows: (see also Table 3.2)

1. Defining — attempts to provide a definition
2. Explanations — teachers explain an idea or procedure
3. Representations — teachers represent ideas and in various ways

4. Working with learners' ideas — teachers engage with both expected and unexpected learners' mathematical ideas
5. Restructuring tasks — teachers change set tasks by scaling them either up or down
6. Questioning — teacher asks questions to move the lesson on

The illustrations in this section were selected based on their link to SCK for teaching early algebra and used to understand the development of this group of preservice teachers' specialised content knowledge for teaching early algebra. Not one of the preservice teachers in the study used definitions in their early algebra lesson so this was excluded from the analysis. The illustrations from the verbal and written responses were categorised using only five of the Kazima et al. (2008) tasks of teaching. The analysis that follows raises important and pertinent issues which are relevant to mathematics teacher education and to the teaching of early algebra. Preservice teachers are required to transform common content knowledge of early algebra into knowledge that is accessible and understandable for learners' i.e. specialised content knowledge (Ball et al., 2008). The illustrations reveal what preservice teachers understand and think about while transforming knowledge in respect of the five tasks of teaching and form part of a reflection on their practice having taught an early algebra lesson. The reflective responses help give an indication of the development of specialised knowledge for teaching of preservice teachers.

4.4.1. Knowledge of explanations

This first category of specialised content knowledge requires teachers to have knowledge of explanations of ideas or procedures and involves designing mathematically accurate explanations that are comprehensible and useful for learners. It entails unpacking mathematical knowledge in order to provide meaning for learners and includes presenting, explaining and justifying mathematical ideas (Ball et al, 2008: 400). The use of mathematical notation and language are also part of explanations and is included in this category of specialised content knowledge.

The illustrations given below indicate knowledge of explanations which involves the use of different representations and the sequencing of tasks. The unpacking of mathematical knowledge for the early algebra lesson followed a similar format in

most instances: engaging in a problematic within a functional situation: creating a record, and seeking patterns and mathematical certainty (Smith, 2008).

The explaining of the extra two people on either side of the tables was also an easy transition with the learners understanding that concept. Up to this point I had been using pictures on the board to explain the concepts and then I moved onto the flow diagram to see if they could move their head knowledge onto paper. My steps were short so they could understand every step that I took. It was a lot easier explaining to them through the flow diagram putting the inputs on the left hand side and the outputs on the right. I do feel that they did have a good understanding of the algebra by the end of the lesson. (S6 - LR)

I would still use the concrete and then step by step using modelling and demonstrations, I would go through the table with the learners. I would then use the information in the table to predict patterns and gather information about the numbers in the table, this would be all that I would include in the lesson for now until the learners have understood the concept of the function table and are able to use the information to predict patterns and make deductions. (S13 - LR)

Some learners also struggled when I handed out the worksheet and it asked for the number of eyes for 100 dogs. The big numbers seemed to throw them off a bit, even when I tried explaining to them that it doesn't matter once they have found the function rule for the other numbers. However, I explained to them using the table, and I should have used the flow diagram first, as they probably would've understood more if I had done that. (S22 - LR)

While the early algebra lesson explanations appear to follow a particular format, there were additional elements that also featured as part of unpacking the mathematics to make it comprehensible for learners. These include linking algebra to real life contexts, establishing a firm foundation of understanding before moving to abstraction (generalisation) and linking verbal and written descriptions to help develop understanding.

My goal upon approaching this class was to show them that algebra is real and not abstract and this is the direction that I followed in my Mathematics lessons – to put Mathematics into context. (S8 - LR)

I also needed to give the learners a more firm foundation before introducing them into the abstract side of the lesson. (S17 - LR)

I realised what I must work on is writing what they are saying on the white board or board behind me for them to actually see the numbers and connection. (S18 - LR)

The illustrations represent three different aspects of explanations which are pertinent for these preservice teachers. Firstly, explanation of the function concept is represented as a series of steps which need to be unpacked in a particular order, starting with the collection of data, representation of the data in a table and/or flow diagram and analysis of the relationships to establish a function rule. Secondly, explanations are presented as links to patterns and real life to provide a context and relevance for learning mathematics. Thirdly, explanations are used as opportunities to make connections between learner verbal and written responses, and to lay a firm foundation of patterns before moving to generalisation.

LMT (2006) developed codes to analyse video recordings of mathematics lessons and distinguished between mathematical descriptions, mathematical explanations and mathematical justifications. Mathematical descriptions (of steps) on the one hand are teacher (by self or co-produced with students) directed and provide clear characterisations of the steps of a mathematical procedure or a process (e.g., a word problem). They do not necessarily address the meaning or reason for the steps. Mathematical explanations on the other hand give mathematical meaning to ideas or procedures. They are still teacher directed although they include attention to the meaning of steps or ideas but they do not necessarily provide mathematical justification. Mathematical justifications are teacher directed explanations which include deductive reasoning about why a procedure works or why something is true or valid in general.

The analysis of the explanation illustrations establishes links between mathematical descriptions in which the preservice teachers present the function concept as a procedure; and mathematical explanations where they recognise the difficulty that learners have in making connections. There is an effort to give meaning to the procedures by connecting the concepts of patterns to manipulatives and real life contexts. However there is little illustration of mathematical justifications which explain why procedures work, or testing for validity.

Teaching someone how to explain is also not easy. Rowland (2012) recognises explanations as central to the task of teaching and acknowledges the difficulty

involved in trying to teach someone else how to explain. He proposes the construct of an “explanation repertoire” which is a mathematical instructional explanation and characterised by one or more of the following: use of representations, examples and analogies; inductive and plausible reasoning; and proof via generic examples (p.64). The preservice teachers’ illustrations demonstrate the beginning of an “explanation repertoire” through the considered use of representations and real life examples. The development of inductive and plausible reasoning can also be linked to the planned sequencing of the lesson with a focus on functional thinking.

Another challenging element of explanations is the selection of an appropriate representation to assist in instructional explanation and learner understanding. The illustration from S22 (above) presents an example in which the preservice teacher recognises the table model for function as a limiting cognitive tool and suggests using an alternative representation. Tall, Mc Gowen and Demarais (2000: 256) propose an expansion of the notion of representation. Instead of having a variety of different contexts from which the learner is expected to abstract, rather use a “generic embodied image” such as a function box (flow diagram) to act as a “cognitive root” for the development of the concept of function. Through this approach, the function box is not seen just as a pattern-spotting device but as a concept that “embodies the salient features of the idea of function” (p.255). The concept of cognitive root and the role of representation will be discussed in more detail in the next chapter.

The second feature of knowledge of explanations, mentioned earlier, involves the use of mathematical notation and language to help make features of particular mathematical content visible to and learnable by learners (Ball et al. 2008: 400). The preservice teachers highlighted various issues of language which they found interesting and demanding during their teaching practice experience.

There are two aspects of language illustrated in the following extracts: the need for precise and consistent use of terminology in the mathematics classroom; and the use of everyday familiar contexts, such as objects or animals. The PSTs felt it was important to be aware of how language is used in the classroom to reduce possible language difficulties and to think about using familiar contexts to introduce the concept of functional thinking.

There were language issues during the lesson, when it came to the concept and was the question about cuts or was it about pieces. I think that the terminology confused the learners a little bit. I don't think that they didn't understand the concept of halving or doubling, I just think that the words used created a barrier which lead to the confusion. (S15 - LR)

I also needed to make sure I am always 100% clear with the learners. I can't say 'chairs' one time and 'people' the next. It might be a silly or unimportant mistake to me, but I think it could throw off some of the learners. (S25 - LR)

Therefore the only solution that I can see is to give the Grade 2's an easier concept to start off with like dogs' eyes because the language used is simpler and easier for them to understand. Then after more exposure to the concepts of data tables etc I could have progressed onto the activity I presented to them during the lesson. (S15 - VQ)

The next set of illustrations cautioned against focusing too much on the mathematical terminology such as input, output and function rule which can distract the learners from the concept of functional thinking. However, some of the preservice teachers admitted that they did not think directly about language and terminology when planning the lesson and focused on development of the concept of function instead. There was also a tendency, in the younger grades, to translate mathematical terminology into simpler, more familiar terms to make concepts more accessible to learners eg. function tables become magic boxes and function rules become recipes.

I tried to include language like function table and inputs and outputs but again this needed more background for the learners to create conceptual knowledge of these terms in order for them to understand the concepts in order to use them in the future. (S13 - VQ)

There were many opportunities to do so but I didn't capitalise on those opportunities, I could have done it in the flow diagram, mentioning inputs and outputs, I also could have put a letter where there was an unknown but I didn't even think about the language, I was more focused on the understanding of the concepts. The only algebraic word which I emphasized was 'relationship'. (S6 - VQ)

When we found what must go inside our 'magic box', I asked them if we had found our recipe that we can use for any number. Every time I referred to it as a recipe, I would also say the word 'formula'. Instead of making a big deal of the new word I simply left it to incidental learning. (S3 - VQ)

The illustrations indicate an awareness of the multiple use of mathematical language in the classroom and emphasise different challenges for the preservice teacher. Firstly, there is recognition of the need to use language consistently to avoid confusion for learners, for example, deciding to use chairs and/or people around a table. Secondly, deciding when to include mathematical terminology such as inputs, outputs and function rule, and lastly using everyday language to represent mathematical language eg. function rule becomes recipe. While it is important to be mathematically precise and consistent, it is equally important for preservice teachers to realise the importance of language in planning explanations that are useful for learners.

LMT (2006: 10) found in their analysis of video recordings of mathematics teaching that "teachers' treatment of mathematical language was a probable indicator of mathematical knowledge and a major aspect of the overall mathematical quality of a lesson". They found some teachers used mathematical terms correctly but were unable to explain the mathematical idea using everyday language, while others did not differentiate between everyday and mathematical meanings for particular words. They devised two separate codes to distinguish between the different meanings i.e. technical language and general language. Technical language (mathematical terms and concepts) is the appropriate use of mathematical terms and includes care in distinguishing everyday meanings different from their mathematical meanings. General language for expressing mathematical ideas (overall care and precision with language) is used to convey mathematical concepts which include analogies, metaphors, and stories when used. It includes the sensitive use of everyday terms when used in mathematical ways (e.g. borrow).

The analysis of the language illustrations shows an awareness of the issues related to technical and general language usage in the mathematics classroom. The preservice teachers highlight the problem of using too many mathematical terms without building the concurrent conceptual understanding. Others use general language to explain mathematical ideas without linking the technical mathematical language, while still others use everyday language to represent mathematical

concepts and tools such as function rule/recipe and function box/magic box but do not distinguish between the different meanings.

Knowledge of language represents specialised content knowledge that is needed to transform common content knowledge so learners can make sense of the mathematical content. It requires careful consideration of the role and use of language in the lesson planning and execution. Functional thinking cannot develop as a result of a set of procedures; language is needed to mediate understanding and to develop conceptual understanding of functions. It is problematic when careful consideration of the role of language in early algebra is ignored or transformed to represent mathematics as a magical process and not the analysis of relationships with the view to generalise.

4.4.2. Knowledge of representations

The second task of teaching looks at preservice teachers' knowledge of representations and how they represent ideas in various ways. There are five elements of knowledge of representations which have been identified and extracted from the work of Ball and her colleagues (2004; 2008). Knowledge of representations is demonstrated when the teacher can:

- identify the task of representing ideas carefully, mapping between a physical or graphical model, the symbolic notation and the operations or process
- make connections between the representations
- recognise what is involved in using a particular representation
- select representations for particular purposes
- make and use representations effectively.

The combined responses from the reflections, questionnaires and focus group interviews are used to illustrate the development of preservice teacher knowledge of representations as used in the early algebra lesson.

Bruner (1966) identified three forms of representations which can be used to develop a spiral approach to learning: enactive (real works and manipulative); iconic (drawings and diagrams) and symbolic (symbols). These are often interpreted in a linear, progressive sequence with the hope of weaning learners off the physical model and encouraging the progression to symbolisation. Representations can

also be viewed as ways of expressing mathematical structures (relationships) and can represent mathematical ideas in multiple ways: manipulatives, diagrams, real life situations, verbal and written symbols (Lesh et al. 2003). The knowledge of representations found in the verbal and written responses is varied and multi-faceted reflecting both the above mentioned aspects of representations and ways of expressing mathematical relationships.

a) Representing ideas carefully mapping between a physical or graphical model, the symbolic notation and the operations or process.

During the lesson, pattern activities were presented physically using objects to manipulate (enactive) or visually (iconic) using pictures or diagrams, to help create and extend a sequence towards generalisation (symbols) of the pattern. The data derived from the physical and visual patterns was usually captured in vertical and horizontal tables and/or flow diagrams depending on the preference of the teacher. The careful mapping between the different forms of representations appears to be guided by the preservice teacher's own prior knowledge, experience and confidence as well as the perceived abilities of the learners to find the function rule.

Most effective part of lesson was starting with real-life situation and also linking up the pictures to the table, the table to the formula, the formulas to the pictures and the output values on the table to the graph. (S8 - LR)

I used the relationship between the tables and people and I also used the flow diagram on the board. I chose these representations because I had a good understanding of them so I felt very comfortable in teaching it. When I am comfortable I can be confident in my teaching whilst if I was teaching a representation I had doubts about I wouldn't be too confident. (S6 - VQ)

I chose to use a table to represent the information because I thought it would be easier for the learners to identify a pattern and the function rules. However, I think that perhaps the spider diagram (flow diagram) may have been better suited for this purpose. (S23 - VQ)

b) Making connections between the representations

Mason et al. (2005) highlight the importance of recognising how representations can be used to help make connections as well as seeing them as different interconnected ways of expressing mathematical relationships.

The illustrations which follow express the connections preservice teachers try to make visible through the early algebra lesson. It starts with making connections within the table model by allowing learners time to think and to make conjectures about the pattern. This is followed by an example in which the preservice teacher moves between different models of functions to show how patterns can be represented in different ways. The last extract looks at real life examples to the table representation in the hope of helping learners make sense of the mathematics.

The learners put this into a table and extended it to the 6th term, or $x = 6$. They then had to figure out the formula for this pattern. I gave the learners a chance to see what is happening in the pattern firstly. They notice the pattern was increasing by three and a few learners even said that they were able to see the formula already. (S1 - LR)

Diagrams, tables and graphs were used as mathematical representations in the lesson. I used this to show how the same pattern can be represented in a different way i.e. a diagram can help us to complete a table, and the values on a table can help us to plot a graph. (S8 - VQ)

For me the question is, in teaching practice when you teach maths, children always ask you why must I do this. I'm not going to use it everyday but when you use it in real life situations I say you are going to use it and this is where you are going to use it and this is how you going to... (FGI)

c) Selecting representations for particular purposes

The illustrations indicate a selection of different representations being used to build an understanding of pattern as function. One example employed by a preservice teacher was to extend the learners' knowledge of the times tables pattern ($3x$ and $2x$), and to investigate other related patterns ($3x + 1$ and $2x + 1$) using a range of input values. Learners were given time to engage with the pattern and then encouraged to identify the function rule.

The use of real-life contexts as well as pictures, especially related to animals, is often used to investigate relationships such as the number of dogs and their total number of feet. This appears to be a popular choice of task followed by tabulation (flow diagram or table) and identification of the function rule. Concrete materials, matches and blocks, are also used to create models, extend and analyse different patterns.

Using the concrete material helped the learners create connections between moving from one term/picture to the next, and they could see the pattern about adding two each time to create the next picture/term. Using the table to represent the data worked well, the learners understood the concept to record their information. (S13 - LR)

Another of the preservice teachers linked patterns to piggy banks and money because she felt the learners would make more sense of this context as they come from impoverished communities where money is a problem.

I chose to use an activity that the learners could relate to in the real world. I chose the concept of a piggy bank and saving money as this could possibly help the learners due to the disadvantaged backgrounds they are living in. The reason behind this is that it was something that the learners could physically see and make sense of. (S24 - VQ)

d) Recognising what is involved in using a particular representation

This preservice teacher believes that learners need opportunity to make sense of problems, and use the concrete apparatus in different ways to extend the pattern and generalise.

The focus of my lesson was identifying the common difference and recognising the pattern and recording information in a table through the matches activity (square) learners had to see, observe and do. Maybe if learners had to first create the 4th, 5th and 6th square with the matches on their own and see if they can identify and pick up a pattern (add 3) maybe this would have been better..... A good idea would have been to let the learners build one block and add matches to the same block to get 2 blocks then 3 blocks then 4 and so on and not doing it separately, they would have seen it clearly and quicker. (S10 - LR)

e) Making and using mathematical representations effectively

These illustrations show recognition of the problem of moving too quickly between different representations i.e. table to flow diagram and vice versa. This caused confusion for learners and impaired the development of functional thinking. The extracts also show that given the opportunity and time to reflect on the process, the preservice teachers began to develop knowledge of what is required to make and use mathematical representations effectively as indicated below.

The flow diagrams would have been better if I had explained that they worked hand in hand with the tables. I forgot about the flow diagram so when I got to the square shape I remembered we needed to do the flow diagram. I should have explained that the flow diagram is important and it's a good link to the tables. (S7 - LR)

However, I noticed that the learners got confused when I showed them how the data in a flow diagram can be put into a table format. This was a new mathematics concept I wanted them to learn, however I think that majority of the class were not really ready for it. I should do more lessons using tables with them for future lessons to better this skill with them (S22 - LR).

When the learners were busy with their activity, many of them struggled with writing the functional rule in words. I should have paid more attention to that. Perhaps a spider diagram would have helped in this regard as the functional rule is clearly stated on the diagram. (S23 - LR).

Overall, the illustrations indicate that preservice teachers know how to represent patterns in different ways: physically, visually, and symbolically (Bruner, 1966). There is also an illustration to indicate knowledge of the usage of representations to help learners make connections between the different forms of the function and to select activities based on the link to real life and the social context. The preservice teachers appeared to recognise that learners need to be given time to make sense of the pattern functions, and too rapid a movement between representations can lead to conceptual difficulty.

The written responses given by preservice teachers raise three important issues in relation to the use of representations in the early algebra lesson: firstly, there is the presentation of ideas moving from the simple to more complex (Bruner, 1966), secondly, the use of representations to show the inter-connectivity between and within forms of representations of functions (Lesh et al., 2003); and finally the selection of representations for specific purposes e.g. using mathematics to represent and solve 'real life' problems. Preservice teacher appear to understand and use representation in different ways which has implications for teaching and learning. The use of representations to develop learner understanding of functions is critical and may influence the strategies that learners adopt to solve pattern generalisations tasks. While the sequential approach to representations may be helpful to some learners, it may create complications for others which could be true

for any of the approaches. However what is important from this is to realise that representations can play a key role in knowledge development for learners.

The transformation of specialised content knowledge into knowledge that is accessible for learners requires the use of representational models. Hart (1989) cautions if teachers do not stress the connectivity between the practical work and the symbolic statement, learners may not understand the concept or the inter-connectivity of the mathematics concepts. These preservice teachers were trying to make sense and establish linkage between theory and practice and in doing so faced many complex issues which emerge as a result of their developing practice.

4.4.3. Knowledge of working with learners' ideas

Within this category, Ball et al. (2004: 59) identify three aspects of working with learners' ideas that are important in transforming knowledge for teaching:

- interpreting and making mathematical and pedagogical judgements about learners' questions, solutions, problems and insights;
- responding productively to learners' questions; and
- assessing learners' mathematical learning and taking the next step.

Each of the three aspects are discussed and illustrations of specialised content knowledge of teaching taken from the verbal and written responses are provided. The analysis focuses on what the illustrations indicate about the specialised content knowledge for teaching early algebra and preservice teachers' engagement in this process.

a) Interpret and make mathematical and pedagogical judgments about learners' questions, solutions, problems and insights.

The PST extracts suggest they found it difficult to work with learners' solutions, interpret and make judgements about learner contributions.

My biggest problem with the lesson was my ability to understand and work with the learners' answers (when they weren't what I was hoping for). I would get stuck and nervous that I was losing the other learners in the class and I was afraid to drift too far away from my planned lesson. I think in the future I need to be more flexible in case my learners don't always think in the same way as I do.
(S25 - LR)

It is imperative to paraphrase what the learners have said in order for the rest of the class to hear and understand in “their” way; although I did this at times, I didn’t do it often enough and perhaps left room for a stunt in algebraic growth. (S24 - LR)

The following are examples of preservice teachers’ awareness and desire to encourage learners to think and explore options for themselves and to work with learner responses in a productive way.

The way I showed them their mistakes and corrected them by doing their sums on the board with them (trial and error) was also a good strategy as they could see for themselves why something would not work, instead of just being told it wouldn’t (S22 - LR).

I also find that being able to explore different ways of thinking allows for more understanding, especially for those learners who don’t ask questions and are too shy, this way with all the different explanations each of the learners are able to find a way of working it out which they find most comfortable. (S26 - LR)

The learners all participate and they all answered relevantly. I used what the learners said to build on what they knew and then use that to introduce the new concept. I then would repeat what was being said that I thought was of good use. (S4 - VQ)

b) Respond productively to learners’ questions

This aspect of knowledge of working with learners did not feature regularly in the preservice teachers’ responses and is partly due to the lack of opportunity for learners to ask questions. Most of the lessons involved whole class teaching and activity based pattern activities which were teacher guided; there were very little independent or group activities in which learners had a chance to discuss or ask questions. There was one instance in which a preservice teacher wrote that when she did not know how to answer the question; she repeated what the learner said and then moved on (S18).

c) Assess learners' mathematical learning and take the next steps

There are a number of illustrations given in both the reflections and questionnaires which link the assessment of mathematical learning and the subsequent effect on instructional practice. Preservice teachers appear to use the opportunity of

assessing learners' mathematical learning to critical reflect on pedagogical improvements needed in future lessons. Some of the issues raised include difficulties with helping learners to construct the function rule, giving time for exploration and self discovery and being too prescriptive within the lesson as indicated in the transcript below:

Learners were not able to identify the rule so I guided them there but I'm not sure it was the absolute correct thing to do but they all did seem interested even though they were not thinking themselves. (S13 - VQ)

I should have given more time for self discovery in which the learners were given the opportunity to explore the problems on their own or in groups before addressing the problem as a class. Children learn through exploration and involvement and need to be given opportunities to make their own conjectures. I should have been less of a teacher and more of a facilitator (S14 - LR)

One of the comments I made after the lesson was that I felt as though I'd been too prescriptive- leading the learners in their thinking rather than exploring their thoughts for fear that following their thoughts would lead to confusion. (S16 - LR)

The preservice teachers' illustrations appear to reflect a particular attitude and orientation towards mathematics which welcomes and encourages learner participation through the sharing of solutions and insights. There is recognition of the usefulness of working with errors, paraphrasing learner responses and allowing learners to discover the mathematics for themselves. The teacher is a facilitator of this learning and does not prescribe learner action. There are strong links to constructivist theory of learning in which the teacher and learner are co-constructors in the production of knowledge e.g. "children learn through exploration and involvement" (S14 - LR).

Empirical research suggests that how teachers work with learners' ideas is often linked to how they hold mathematics. Barton (2009: 7) identifies a missing component of mathematical knowledge for teaching i.e. "how a teacher holds mathematics" which refers to the teacher's attitude and orientation towards mathematics. It comprises four parts: a teacher's vision of mathematics, a teacher's philosophy of mathematics, a teacher's sense of the role of mathematics in society and a teacher's orientation towards the subject. How a preservice teacher holds mathematics impacts on their actions in the classroom and more especially on their

work with learners. Their instructional actions are guided by their beliefs about mathematics (vision), and their beliefs about teaching and learning (philosophy and orientation).

The responses given by the PSTs appear to reflect a conflict in their role as teacher and in their beliefs about how learning happens. They recognise the difficulty learners have in identifying the function rule but are cautious to become directly involved in the lesson. There is an awareness of their own shortcomings such as accepting all learners' contributions, lack of willingness to deviate from the lesson plan, too much teacher guidance, not enough time for discovery, and too few learner questions.

Fennema and Franke (1992) argue that knowledge for teaching comprises four elements: knowledge of content; knowledge of pedagogy; knowledge of learner cognitions as well as teacher beliefs. Knowledge is seen as interactive and dynamic and subject to change as teachers participate in different experiences. The extracts from the reflections and questionnaires highlight the difficulty preservice teachers experience in working with learners' ideas and transforming knowledge and teaching to include a more interactive and participative approach. Their knowledge and understanding of mathematics and mathematics teaching is evolving through their interaction with the learners. Their beliefs about learning influence their decisions about instruction and the teaching of the early algebra lesson. They want learners to be more involved in the individual construction of knowledge but are not always sure how to manage the situation.

4.4.4. Knowledge of task design

The fourth task of teaching, is making judgements about the mathematical quality of instructional materials and modifying as necessary (Ball et al., 2004: 59). This task was later elaborated to entail the following (Ball et al., 2008: 400):

- appraising and adapting the mathematical content of textbooks
- modifying tasks to be either easier or harder
- finding an example to make a specific mathematical point.

The instructional materials used by the preservice teachers were usually self generated and adapted from textbooks and internet sources. The illustrations given below reveal their responses having taught the early algebra lesson and reflected on

the use of tasks and examples in the lesson. The extracts pertain to modifying tasks and finding examples only as there were no responses related to appraising and adapting textbooks material.

a) Modifying tasks to make them easier or more difficult

As mentioned earlier the illustrations given below refer to a previously taught early algebra lesson therefore the modification of tasks is done in hindsight and emphasises possible suggestions for future lessons. It starts with the need for planning for connectivity so learners can make conceptual links between tasks. It also includes planning for progression in tasks to move from easier to more difficult tasks and to include extension tasks to accommodate different levels of students.

Next time I should think of activities and how they connect and lead up to next activity so learners can understand better. Activities should have been more basic for the type of class I had, I should have done triangular match activity then square matchstick activity (S10 - LR)

I would have planned my lesson in such a way as to accommodate the high flyers as well, I would have designed worksheets that covered every level in the classroom, for example, questions everyone can answer, questions some could answer and questions only the top achievers could do. (S11 - LR)

I limited the lesson somewhat, in the sense that I did not plan for much extension with the work I was teaching and so I would recommend that various options should be kept on hand just in case you need that little bit more, or less depending on the situation. (S26 - LR)

The next set of illustrations focus on the assumptions these preservice teachers made in the initial planning of class tasks on functional thinking and explains how they modified the tasks to respond to the teaching context. The one preservice teacher assumed that learners would move easily to generalisation and ended up modifying the task by giving the output values and scaffolding the function rule. Another found it necessary to 'undo' the mathematics by giving the output values and the function rule and asked for the input value. The last preservice teacher realised that the model selected to assist the explanation might have hampered understanding and realised that an alternative representation might have been more useful in the completion of the task.

I assumed that as soon as somebody said "times 2" they would see that that was correct and they just needed to add the plus one. During the lesson however, this did not happen. Generalising the formula was extremely challenging for them. To guide them, I wrote the answers and asked them to find the relationship (what more do we need to get to our number? (output)). (S3 - LR)

Near the end of the lesson I swapped the inputs and outputs around on the flow diagram. I asked them "How many tables would there be if there was this many people?" This truly did test their understanding and a lot of the learners could give the correct answers but some found themselves having more tables than people which they noticed and then correcting themselves but swapping the function rule around. (S6 - LR)

During the exposition I observed that the Grade 2's struggled with the horizontal data table concept at first, perhaps if I had given them each their own table to fill in it would have been more relevant and made more sense to them. Looking back now I could have also first started with the flow "spider" diagram as it was easier for them to see how the inputs and outputs related to one another in this way than through the use of the table. (S15 - LR)

Next time I will use counters as well for the learners who are weaker and along with my table I will use cut outs so they can more easily visualise the outfits. (S17 - LR)

b) Finding appropriate mathematical examples

Finding a suitable example to introduce the patterns lesson can be difficult for some preservice teachers as they lack the relevant experience and confidence in teaching. The following illustrations present two different perspectives in terms of finding appropriate examples. Both extracts emphasise the need for concrete and practical experiences to help grasp the concept of functions. However the second illustration shows a more developmental approach to the overall planning and execution of the lesson.

I had no idea what I could do as an introduction and as a conclusion to this lesson. I phoned other students and asked them for help....this is basically what influenced my selection and sequencing of my activities. I did however choose the string activity, because I realised that it had potential to become a very concrete activity which would help the learners in grasping this brand new concept. (S3 - VQ)

I really liked the introduction of the lesson, the body percussion, as it introduced patterns and beats to the Grade 2's in a fun and exciting way. The exposition of the lesson, cutting wool, was practical and made it easier for the Grade 2's to see the pattern being dealt with. The table used in the exposition also further helped the Grade 2's grasp the pattern. During the exposition I also introduced algebraic symbols by getting the Grade 2's to see a shorter way of writing the information in the table i.e. instead of writing out the whole word cuts and pieces they could just write a "c" and a "p".(S15 - LR)

The preservice teacher illustrations above show development of knowledge of task design both in modifying tasks and finding examples to make a mathematical point. They draw attention to issues of planning which are challenging for preservice teachers but they also reveal an understanding of task design. The PDT's are able to highlight essential features of task design such as making links between activities, planning for progression and catering for differences in difficulty as well as the selection of appropriate forms of representations. They are aware that task design is a critical part of concept development but that it is not easy to appraise, adapt or modify tasks without the relevant knowledge or teaching experience. Their illustrations outline some of the decisions that preservice teachers have to make in terms of what is complex or conceptually appropriate for the class. However, they also expose the integrated nature of specialised content knowledge for teaching and the difficulty of dealing with each of the tasks of teaching in isolation. It is complicated to design tasks without thinking about the learners, the mathematical concepts and skills, the explanations needed and the appropriate representations. This is a big challenge for preservice teachers who are beginners in this process.

4.4.5. Knowledge of questions

The final task of teaching is questioning and involves asking good mathematical questions and posing problems that are productive for learners' learning (Ball et al., 2008: 400). The preservice teacher responses in this category are drawn from the lesson reflections and the focus group interviews only. There was no direct link made between this category and the questions in the video questionnaire. The illustrations which follow demonstrate an important awareness of both the strengths and weaknesses of the questioning ability of these preservice teachers.

The following extract gives an illustration of a preservice teacher who felt confident in asking questions which lead to a particular outcome. There is an emphasis on

sequential learning through guided questions moving from pictures of patterns, tabulation of values, construction of function rule to plotting graphs. While there is a focus on ways of connecting different representations of functions, there is no reference to high order questions that require learners to make conjectures or to give explanations of answers.

I think that the questions used in the lesson were good and related to what was done in the lesson such as asking them to extend the picture and to add the values on a table, in order for them to arrive at a formula and also to plot the values on a graph. (S8 - LR)

The following illustrations indicate an awareness of the difficulty in the use of questioning as a pedagogical strategy. They show that these preservice teachers recognised their lack of knowledge of questioning especially with regard to using too many prompting, leading and guiding questions, and/or asking poorly phrased and ambiguous questions. They were aware of their inability to pose and probe learning through effective questioning but they could suggest possible remedial action.

My questions could have been clearer, should have had more questions beforehand. I guided the learners too much. I didn't give them enough room to think on their own, to explore with maths. (S5 - VQ)

I felt that my questions weren't well thought through so the learners battled to understand what I was asking at times. This forced me to walk through every step with them with me doing a lot of the thinking for them rather than them understanding it themselves. I found myself asking "do you understand?" rather than me asking "what is your understanding?" (S6 - LR)

I realised that my questioning techniques need to be on a much higher level because if they knew something or if I didn't know something, I didn't know what question to ask them to bring them to that answer and that's what I found that in any lesson, algebra or in any maths lesson especially my questioning techniques needs... somehow I need some help in doing that because the learners, its not that they didn't know but I didn't know how to ask them the right questions for them to give me the right answer. (FGI)

These preservice teachers felt they did not give learners time to explore the patterns, asked too many questions and did not follow learner explanations with

probing questions. There was difficulty with the level and clarity of the questions which caused confusion and was not rectified in rephrasing.

.....the level of the questions, the clarity of the questions that were asked - here these questions needed to be clearer in what I was asking the students - I would present a question then get no response from the learners so would change the wording of the question once, twice if not three times and this caused confusion and the lesson became very bitty and up and down. The level of the question in terms of challenging the students more, could have been improved. (S12 - LR)

I think the lesson that I did with them was up until a certain level but I didn't know how to take it further. I didn't know that's what I'm still trying to think exactly what to say, like I didn't know...if they said something I didn't know how to take it and say okay so you say this, so is she right, is she wrong, question her because I was at a girls school. Like question her. The questioning techniques, and the modelling techniques. I was very inexperienced. (FGI)

Notwithstanding the difficulties of posing good questions, there were a number of interesting and useful suggestions given by the preservice teachers to indicate a growth of understanding of specialised content knowledge for teaching patterns and functions. Suggestions included the need to pay attention to learner responses and to capitalise on their explanations. They felt that it was important to ask a variety of questions, both open and closed, and to probe and extend learner thinking because poorly phrased questions can create further problems of learners.

While preservice teachers acknowledged the difficulties of questioning and they also recognised that fluidity of questioning comes with experience.

I should also be more aware of the questions I ask and the answers I receive, and show more interest in the learners responses and ask "how they get their answers, so as to get some insight in where the learners are and if they are making the connections and discovering patterns". (S13 - LR)

Posing extension questions could raise the level of understanding of the concept. (S12 - LR)

Asking meaningful questions that lead to higher order thinking was a challenge, something I feel I'd have to work out before the lesson because while the lesson is happening, the questions don't come naturally. (S16 - LR)

Developing effective questioning skills is an essential part of teacher education and requires a shift in the practices and beliefs of those involved in the interaction (Moyer & Milewicz, 2002). Ralph (1999a, 1999b) propose four basic oral-questioning skills (among others) which are essential for effective questioning in mathematics: preparing important questions ahead of time, delivering questions clearly and concisely, posing questions to children that stimulate thought, and giving children enough time to think about and prepare an answer (in Moyer & Milewicz, 2002: 296). The preservice teacher illustrations reveal that effective questioning is a challenging teaching task which results in poorly worded questions that do not help to develop learners' thinking. Preservice teachers need practice in preparing and delivering different levels of questions to stimulate learner thought while allowing time for learners to prepare answers.

The use of open-ended questioning involves managing additional and unexpected responses from learners and can be complex for the preservice teacher (Moyer & Milewicz, 2002). Experienced teachers usually have the resources and pedagogical content knowledge to draw from and use in the classroom situation. Rosu & Arvold (2005) found that through questioning, preservice teachers began to inquire about their own teaching and mathematics knowledge and begun to “generate theories and practices of questioning to help their learners”. The researchers realised that what was initially seen as a dichotomy i.e. questioning as mathematical development and questioning as skills development became entwined and this helped preservice teachers to inquire into both their teaching skills and their mathematics knowledge.

There is little doubt as to the complex nature of questioning and it is evident from the illustrations that preservice teachers experience similar issues. However the extracts also show that having preservice teachers think about questioning helps develop their knowledge of this task both in terms of their questioning skills and their mathematical knowledge. Through the teaching and reflection of the patterns lesson, they became aware of the vital role of questioning. The experience also appears to have had a ripple effect in that it impacted on their beliefs about mathematics teaching and learning and the task of questioning in this construction.

4.4.6. Summary of preservice teachers' SCK of early algebra

Specialised content knowledge (SCK) is the mathematical knowledge that “allows teachers to engage in particular teaching tasks” (Hill et al., 2008: 377). Teachers require knowledge of explanations and language; representations, working with learners' ideas; task design and questioning (Kazima et al., 2008). The illustrations provided by preservice teachers through their reflection on the early algebra lesson and their conceptual understanding of functions demonstrate a shift in the purpose of mathematics teacher education. It is a “move away from a teacher training paradigm” of teaching content knowledge and pre-defined skills towards a “teacher education paradigm of developing educational knowledge in mathematics” (Bergsten, Grevholm & Favilli, 2009: 67). There is a need to understand teacher education as part of the continuum of knowledge development, in all its facets, which continues to grow and develop during the life cycle of the teacher.

The illustrations provided specify preservice teachers' developing knowledge of particular teaching tasks. Their responses indicate that explanations involve teacher directed lessons without necessarily giving priority to mathematical meaning or justification to the steps or ideas. The preservice teachers select real-life or familiar contexts to make the explanations more real and accessible for learners. They use explanations to make connections and to lay the foundation for generalisations. Language is used in explanations either in a technical way or generally to express mathematical ideas but there is little or no explicit talk about the meaning and use of the mathematical language of functions (LMT, 2006).

The preservice teacher knowledge of representations is illustrated through the selection of concrete apparatus, diagrams, and pictures to represent the idea of a function which are appropriate for the learners, and for the development of the mathematical concept. They try to build connections between the different representations but do not always make the links mathematically explicit e.g. discussing the connection between verbal and written generalisations. The representations become tools in the development of the functional thinking and opportunities are lost to use the models as cognitive roots.

The task of working with learners' ideas involves eliciting and responding to learner descriptions of procedures and ideas and explanations of solutions, interpreting learner productions and using learner errors (LMT, 2006). The illustrations reveal a

dilemma for the preservice teacher who is involved in a process of change or “epistemological transition”. Winslow et al. (2009) highlight the complexity of the development of professional knowledge which teachers need to teach in desirable ways. Even if the teacher has sufficient “academic mathematical knowledge” of the curriculum, the lack of a “coherent, corresponding and articulated pedagogical organisation of knowledge” leave few possibilities to avoid an instrumental approach (p. 94). While the preservice teachers understand the epistemology of constructivism and seek to execute this practice, it is sometimes difficult in the reality of the classroom. Learner responses, descriptions and solutions are not always easy to predict or work with and can lead to guided and descriptive instruction.

Knowledge of task design requires teachers to be able to plan tasks which are appropriate and enable learners to work productively, and for teachers to be able to convey mathematical tasks or problems to learners (LMT, 2007). The responses from the preservice teachers indicate awareness of the activities underlying functional thinking: engaging in a problematic within a functional situation; creating a record and seeking patterns and mathematical certainty (Smith, 2008: 143). They are also able to identify some of the difficulties in task design (differentiating activities, connecting activities, using models such as the table diagram) from their teaching practice experience. Tasks serve a dual purpose for mathematics educators, firstly as a “means and content by which learning is facilitated” and secondly as a means to facilitate learning through the “reflective process” of design, implementation and modification (Zaslavsky, 2009: 110). Deliberations around task design draw attention to subtleties that preservice teachers are not aware of before their teaching experience, including ambiguities they transmit to learners (Zaslavsky, 2009). They raise both mathematical and pedagogical issues and illustrate the complex nature of specialised knowledge for teaching.

The final task for teaching namely questioning is a particular challenge for preservice teachers which they express explicitly in their illustrations. They are able to identify weaknesses in their questioning ability in terms of the levels and types of questions needed to extend conceptual understanding. They are prone to prompt learner thinking and struggle to use questions which are fluid and coherent. There is recognition of the need for better planning of questions and for different levels of questions, and appreciation for the role of experience in developing this knowledge.

The opportunity to teach an early algebra lesson and to reflect on this process, in the form of verbal and written responses, provides an invaluable learning opportunity for preservice teachers. While much progress has been made in developing a framework for mathematical knowledge for teaching, there is still work needed to determine “what kinds of learning opportunities effectively help preservice teachers to develop such knowledge” (Thanheiser et al., 2010: 3). The illustrations provided by preservice teachers begin to elucidate their engagement in the development of their own understanding, facilitated by opportunities to discuss and reflect on the theory and practice of the early algebra course. There are visible connections and overlaps within the teaching tasks and the knowledge domains. However, the results of this analysis begin to show that preservice teachers are not only aware of “what the mathematics content should be” but also “how the content should be taught” (p.12).

4.5. Specialised content knowledge (SCK) in teaching an early algebra lesson: video lesson examples

The previous findings, both of CCK and SCK, provide an overview of the verbal and written responses of preservice teachers as a result of the Maths 2 course and teaching practicum.

The second stage of the analysis relates to *enacted* knowledge for teaching early algebra as observed in the video recordings of the early algebra lessons (Ball et al., 2008). It was an opportunity to investigate the practice of preservice teachers – the actual work of teaching – to uncover its mathematical demands (Ball et al., 2004). Thames et al. (2008: 2) suggest that teaching primary school mathematics is “highly mathematical work” which requires substantial “mathematical knowledge and reasoning”.

This analysis looked specifically at the specialised content knowledge of early algebra and how this was enacted in the early algebra lesson. The analysis of the video recordings was again guided by the work of Ball et al. (2008) and their definition of specialised content knowledge (SCK). There were different teaching tasks which could be identified in the lessons selected for analysis, such as giving explanations, using representations, questioning, restructuring tasks and working with learners’ ideas (Kazima et al., 2008).

Five video lessons were selected through systematic sampling to represent the continuum of teaching early algebra from Grade 3 to Grade 7. The selection was based on the variety of teaching tasks in the lesson as well as the mathematical demands and opportunities that arose during teaching. The purpose of the video analysis was to investigate practice, the actual work of teaching, to identify the teaching tasks that were prevalent in the lesson and to highlight the mathematical demands and opportunities these tasks presented for the preservice teachers.

Each analysis of the lesson starts with an overview of the lesson outcomes and a short summary of the lesson. This is followed by selected episodes from the videoed lessons which are linked to SCK for early algebra and the tasks of teaching pertinent to the particular lesson. It concludes with a summary and discussion of the key features in the lesson as they relate to SCK.

4.5.1. DANI – GRADE 3

a) Lesson outcomes (as given in lesson plan)

The learning outcome requires the learners to be able to recognise, describe and represent patterns and relationships as well as to solve problems using algebraic language and skills. The purpose of the lesson is to describe observed patterns and to find the relationship between the numbers (pattern) and generate a formula.

b) Lesson summary

The lesson starts with making sound patterns using hands. It then moves to identifying different examples of patterns: number and geometric. The lesson then focuses on the investigation of a particular pattern using string and cuts. The teacher has prepared two diagrams side by side on the board: a vertical table with two columns headings, number of cuts and number of pieces of string and a flow diagram labelled “The magic box”. She reminds the learners this is a similar lesson to the dog pattern lesson of the previous day. The pattern task involves learners making cuts using different pieces of the same length of string and generating the following data: 1 cut \rightarrow 3 pieces of string; 2 cuts \rightarrow 5 pieces of string, 3 cuts \rightarrow 7 pieces of string and 4 cuts \rightarrow 9 pieces. The information is recorded systematically in the vertical table and the learners are asked to describe what is happening. The teacher and learners focus on the recursive pattern +2 generated from the output values after which the learners are reminded again of the work from the previous day which involved trying to find a magic formula to describe the change from input

to output value. The teacher then moves to the other section of the chalkboard and records the same data in a flow diagram labelled “The magic box”. The learners are required to describe the relationship between the input and output values in the form of a magic formula (function rule) using a guess and check technique.

c) Specialised content knowledge

Specialised content knowledge is the mathematical knowledge and skill uniquely needed by teachers in the work of teaching. It involves a number of everyday tasks of teaching, some of which are: giving explanations; asking questions, selecting representation, modifying tasks, and working with learners’ ideas (Kazima et al., 2008). These tasks have been used to help guide the analysis of the early algebra lessons, to identify tasks that were prevalent in the lesson and highlight the mathematical demands and opportunities these tasks presented for the preservice teachers. The findings were used to reflect on the specialised content knowledge as demonstrated by the preservice teachers in the middle of the Maths 2 course.

This lesson shows the preservice teacher engaged in a number of different teaching tasks which involve introducing the topic of patterns, explaining the pattern problem, demonstrating the pattern task, recording pattern data using different representations, questioning to develop understanding and working with learners to describe pattern relationships. There is a clear sense of pedagogical intent in the lesson and evidence of thoughtful sequencing and planning. There is opportunity for the learners to be actively involved in the lesson and the content of the lesson is presented logically and sequentially. The pattern task is designed to develop the learners’ relational understanding of patterns and functions using concrete objects, tables and flow diagrams. The sequencing of the lesson engages learners in a physical activity in which they have to identify two or more quantities that are changing with a focus on the relationship between two variables (Smith, 2008).

The analysis of this lesson focuses on two specific events which occur in the lesson and demonstrate the challenges preservice teachers face and the knowledge needed for successful teaching of functional thinking. The selected examples are not evaluative of the preservice teacher but help elucidate the knowledge for teaching demands faced by preservice teachers.

Working with learners' responses

Dani has created a record of the input and output values in a function table using physical objects (string and scissors) up to this point of the lesson and is progressing towards analysing the relationship (Smith, 2008).

Number of cuts	Number of pieces
1	3
2	5
3	7
4	9

Figure 4.1: Preservice teacher's illustration of a function table

She works with learners to identify and analyse regularities and change in the patterns (DBE, 2011a) and acknowledges and works with the learner contributions even if they are not what she wants.

- Teacher: Okay, Aziza what's happening.?
- Learner: The number of pieces are all the odd numbers.
- Teacher: Ah, Aziza says these are all the odd numbers okay so 3, 5, 7, 9. So, what would come next?
- Learner: 11
- Teacher: And then?
- Learner: 13.
- Teacher: Well done Aziza, you said these are all the odd numbers.

The teacher realises this classification of the sequence as odd numbers will not help her to move the learners towards describing the relationship between the input and output values and encourages the learners to look for a different pattern.

- Teacher: Who can see a different pattern? What else is happening with those numbers?
- Learner: The one side is increasing in ones.

- Teacher: What are we doing from this number to get to that number (pointing to the output values), to get that number, to get that number? Aqeel.
- Learner: Plus two.
- Teacher: We are plussing two each time.

Number of cuts	Number of pieces
1	3
2	5
3	7
4	9
5	11
6	13

Figure 4.2: Preservice teacher’s illustration of a function table

Dani ignores the opportunity to develop co-variational thinking by simply concentrating on the recursive pattern within the output values (Smith, 2008). This provides problems later when the learners are not able to find the ‘formula’ using the magic box.

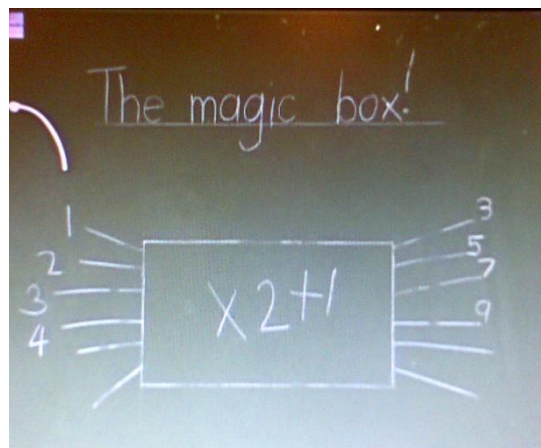


Figure 4.3: Preservice teacher’s illustration of a magic box (flow diagram)

- Teacher: But remember what we did with the dogs? I told you. But what if I saw 700 dogs, how many eyes would there be? Remember we got a formula and a formula was like a recipe and we could use it for every number to see how

many eyes we could get. Do you remember? Right so now we need to do the same thing. What are we doing to these numbers to get to these numbers and how did we do it. What did we use? We used our?

Learner: A magic box.

Teacher: We used our magic box. So we know that when we cut once we got 3, when we cut twice we got 5, when we cut three times we got 7, when we cut four times we got 9. Right what on earth is happening inside my magic box. Ryan, what's happening in there?

Learner: +2

Teacher: Okay so Ryan says we must +2. Okay let's see +2. What is one plus 2?

Learner: Three.

Teacher: What is $2 + 2$?

Learner: 4

Teacher: Okay, so that's not working. Let's see the next one. What is $2 + 3$?

Learner: 5

Teacher: Do you think it's +2?

Learner: No

Teacher: Good try Ryan but not working. Thomas?

Learner: Times by four.

Teacher: Okay, let's times by 4. $1 \times 4 = ??$

Learner: 4.

Teacher: No, something is happening here but I don't want you to guess grade 3s because we could be sitting here forever. I need you to think what could be happening here. Layla?

Learner: Plus 3.

Teacher: Plus 3 but what is $1 + 3 = 4$.

Learner: Plus 3.

Teacher: Plus 3 but what is $1 + 3 = 4$. Kim.

Learner: Times two.

Teacher: Okay let's go with what Kim is saying. Lets x2. Okay what is $1 \times 2 = 2$. And what do we have here?

Learner: Three.

Teacher: But listen so it's not x2 so Kim gets the answer of 2 okay.

What is $2 \times 2 = 4$

What is $3 \times 2 = 6$.

What's $4 \times 2 = 8$.

What's $5 \times 2 = 10$. I'm helping you here. I'm giving you a very good clue.

Nosipho.

The preservice teacher recognises that the ‘guess and check’ technique to help find the relationship between the values is not working and decides to give the class a ‘clue’. Through the act of giving the clue, Dani tries to change the nature of the investigation of the task towards finding the function rule. However, the learners are then no longer involved in studying change and looking for relationships and the opportunity to develop functional thinking is lost.

Finding appropriate mathematical examples

Dani decides to conclude her pattern lesson by encouraging the learners to generate their own pattern and providing the opportunity to “make sense mathematically” (Mason, 2008: 89).

- Teacher: Okay, one more thing you have to do. One more thing, okay. You have to turn your page around and this was my number pattern can you see? Can you see what my number pattern was 3, 5, 7, 9, 11, 13? I want you to write your own number pattern but I don’t want 2, 4, 6, 8. I don’t want 5, 10, 15, 20. I want something different. I want you to write your own number pattern. No, I don’t want 3, 6, 9, 12. I don’t want 5, 8, 12. I want you to think of something different. What else could you do? What could you do to each number?
- Learner: Twenty plus twenty and two hundred plus two hundred.
- Teacher: Okay you could do that or you can do the magic box. You can do the magic box if you want to. It is the same thing as making a number pattern.
- Learner: Can I draw a pattern?
- Teacher: No, no I don’t want you to draw a number pattern I want you to give me a number pattern. Like this. That is a picture pattern, a good picture pattern but I want a number pattern.

The teacher recognises that the learners need to develop their own thinking and understanding through the pattern activity but there is little structure provided to encourage learners to make generalisations, to explain and justify their generalisations and to convince one another (Mason, 2008). The learners need to be helped to move beyond analysing single variable data to two or more quantities simultaneously and to use patterns to make connections and describe relationships at a level which is appropriate for the learner.

4.5.2. ZOE – GRADE 4**a) Lesson outcomes (taken from lesson plan)**

The identified outcome of the lesson is for learners to be able to recognise, describe and represent patterns and relationships, as well as to solve problems using algebraic language and skills. The purpose of the lesson is to investigate and extend numeric and geometric patterns looking for a relationship or rule, including patterns represented in physical or diagrammatic form and to determine output values for given input values.

b) Lesson summary

The lesson starts with a discussion and exploration of different types of patterns: number, shape, colour and people, and moves on to growing patterns. Zoe uses cut out circles to represent a growing caterpillar on the board, starting with two circles on Day 1, growing to 5 circles on Day 2, and 8 circles on Day 3. Learners are asked questions about the pattern and to predict the total number of circles needed for each new day. They have to make, in pairs, their prediction for the total number of circles needed for Day 4 which is then checked and verified on the board using the cut out circles. They proceed to drawing the pattern for Day 5 and compare different solutions. Zoe then challenges the class to predict how many circles will be needed in Day 10 and to justify their answers. The lesson concludes with the teacher helping the learners through a guess and check method to finding a rule to generalise the pattern.

c) Specialised content knowledge*Using representations to support student learning*

Zoe's lesson involves an exploration of patterns which progress from step to step (growing pattern). She demonstrates the physical pattern and motivates the learners to begin to describe the relationship between the number of days and the total number of circles. In an earlier part of the lesson, the learners extend the pattern using the physical objects (circles) and then draw pictures of the next step in the sequence.



Figure 4.4: Preservice teacher's illustration of a table

At this stage, the learners focus on how the pattern changes from step to step, better known as recursive pattern, to solve the problem of the total number of circles needed to represent Day 5 (Smith, 2008; Bezuszkza & Kenney, 2008):

Teacher: Lerato, you can just stand that side quickly. I want you all to look to the front and we want to see how Petra got to her answer. Petra is going to explain to us how she got to her answer.

Learner: On Day 1, I had 2, Day 2, I had 5, Day 3, I had 8, Day 4, I had 11, Day 5, $11 + 3 = 14$ so Day 5 = 14

Teacher: Thank you very much. Thank you , Petra. Did anyone else get that same method, same way. Now let's see what method Lerato got. Let's all watch. Let's see maybe they got a different method. They worked together that's why they have both their boards there.

Learner: We added three circles.

Teacher: So you also just added another 3 circles.

Learner: We added all our circles. We said $2 + 3 + 3 + 3 + 3 = 14$.

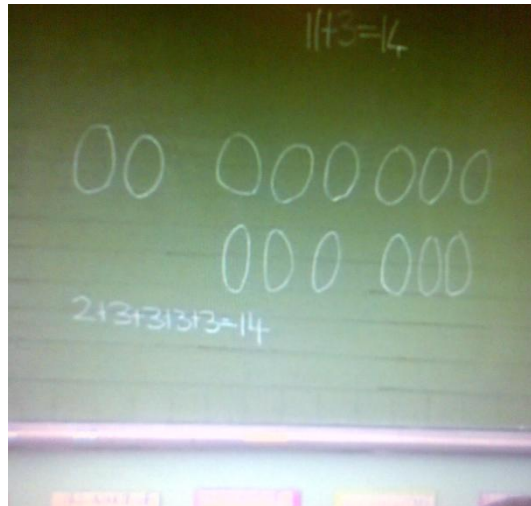


Figure 4.5: Learner illustration of a solution strategy

Zoe then moves to represent the function in a different way introducing a flow diagram to capture the number of steps (days) and the number of objects in the step (total number of circles) which exposes the learners to a different way of looking at and thinking about the function (Van de Walle et al., 2007). This indicates that Zoe is aware of the different representations of functions: physical/context and table/diagram which can be used to illustrate the same relationship and help learners to make connections across representations (p.280). She represents the function idea carefully, mapping between a physical model and the flow diagram and making connections among the representations (Ball et al., 2004). Once the flow

diagram is complete with the input and output data for Day 1-5, she asks the class to predict how many circles will be needed for Day 10 which encourages learners to make conjectures about what would happen for the unknown state (Blanton & Kaput, 2005b).

She shows evidence of selecting representations for a particular purpose, firstly to extend the pattern using the physical objects and moving towards generality using the model of the flow diagram (Ball et al., 2008).

Working with learners' responses

Zoe uses the data from the flow diagram to highlight the difference (recursive nature of the function values as +3) saying “*we’re adding 3 the whole time. Do you see that? We’re adding 3 the whole time*” and then asks the class to guess how many circles would be needed for Day 10 if the pattern repeats in the same way. While the teacher emphasises the difference between the function values i.e. recursive pattern, the learners provide possible solutions to the problem of the number of circles needed for Day 10. They provide three different responses which show they are beginning to make various connections between the different representations of the physical objects and the flow diagram.

Learner 1 (Yaseer) explains that if Day 5 has 14 circles then Day 10 has 28 circles (double the no of days)

Teacher: How long Yaseer? Damion.

Learner 1: 28.

Teacher: How did you get to 28?

Learner 1: I doubled the 14.

Teacher: You doubled the 14. Are you sure? Come up here quickly Damion. Explain to them what you told me now.

Learner 1: Each five days are 14 so if I take another five days it’s another 14 and I double the 14.

Teacher: So you double the 14. Okay. Do you think we should test it? I understand what you are saying but you must remember you must have 3, constant 3 all of the time. So if we have day 6.....

Learner 1 does not refer to the constant difference or the relationship between the days and circles and instead sees the problem as a ratio problem. Learner 2 interrupts the teacher and says that if you are adding +3 each day then 10 days will need $9(3s) = 27$

Learner 2: Miss, I know it's 27, it's 10 days that we have and it's because you counting in 3's.

Teacher: You're counting in 3's.

Learner 2 appears to recognise the pattern of adding 3 for each day but neglects to include the circles from Day 1. Learner 3 alternatively suggests that if Day 5 = 14, so Day 6 = 17, Day 7 = 20.... But does not finish....

Learner 3: It's 28.

Teacher: Is its 28, why? Because you plus 3 all the time. Yes Shuaib.

Learner: I added 6 to the 14.

Teacher: You added 6 to the 14. So that is for day...

Learner: 6.

Teacher: For day 6.

Learner: Day 6, I added 3 and Day 7 I added another 3 and that is 20.

Teacher: Twenty. So I ask on Day 10 how long the body is going to be? It's close. So now I want to show you something. Remember how I said?.

Zoe acknowledges each of the contributions and tries to work with the learners to extend their thinking and realises that Learner 3 is still emphasising the recursive pattern. At this stage, she decides to proceed with her planned instructional goal which is to generalise the function in words and/or symbols and brings the learners' attention back to the flow diagram. While Zoe engages with the learners' various responses, she appears to have difficulty in responding productively to their solutions and to unpacking the mathematical knowledge in order to help learners to make meaning (Ball et al., 2008).

There is no single way of finding a relationship between day (input) and total numbers of circles (output) and learners will probably see it in different ways (Van de Walle et al., 2007). However from the presentation given so far in this lesson, learners have two available options. They can make use of the common difference between the dependent variables and count in 3s until they reach day 10 or they can begin to look for a relationship between the input and output values. From the explanations given by the learners for the number of circles needed for Day 10, learners appear to find it easier to extend patterns numerically even when the patterns are presented visually (Zaskis & Liljedahl, 2002; Moss & Beatty, 2006).

Explanation of functional relationship

While the lesson has progressed logically to this point, Zoe decides it is time to introduce an alternative representation of the function and introduces the use of symbols to describe the functional relationship between the days and the number of circles. She has not resolved the issue about the number of circles needed for Day 10 but decides that finding an explicit formula to describe the relationship is now appropriate.

She reminds the class of an earlier episode in the lesson in which she used letters to represent colours and writes d (= days of the week) in the centre box of the flow diagram:

Teacher: Now remember how I said at the beginning a letter can be in place of number, remember that. Now I'm going to put the letter d . Why am I putting the letter d there? Yes, Shahied.

Learner: For the days.

Teacher: For the days. Yes....

Having inserted the d in the box, she indicates that it represents the number of each day:

Teacher: See the d over there for the day. Now this d can represent either day 1, or day 2, or day 3, or day 4 or day 5. So it can represent any number of days.

She then reminds the class of the $+3$ on the outside of the output values, puts 3 in front of the d ($= 3d$) because there is a 3 difference between the values and then checks if the output is correct:

Teacher: Can you see we have a constant 3 there. (pointing to the $+3$ in the flow diagram). So I'm going to put a 3 over there ($3d$) and that means it's 3 times the day because every time we're adding the 3.

Using trial and error and substitution of $d = 1$, Zoe gives the formula as $3d - 1$ and then checks using $d = 2$ and $d = 3$. She finishes by substituting $d = 10$ and verifying the answer of 29 as correct.

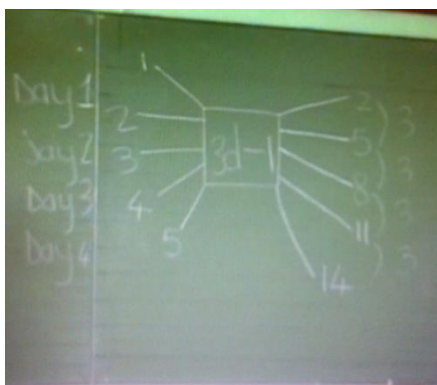


Figure 4.6: Preservice teacher's illustration of a flow diagram

- Teacher: So 3×1 is 3? But that's not the answer we got over there is it? (pointing to the output value of 2). So what do we need to get to that answer? What must we do?
- Learner: Minus one (Zita then writes $3d - 1$ in the middle box between the input and output values)
- Teacher: (pointing to Day 2): So 2×3 is 6 – 1 is 5. So do you see there? What happened? We found a formula to get to our answer. So if I have on the 10th day... now let's check it. So for 3 on the 10th day we must times it by 10 (3×10) is 30 – 1 is 29. So those who said 29 was correct. Good.

The teacher tries to scaffold the learners' understanding, however, the shift from recursive patterning to describing corresponding relationships is poorly managed and becomes a routine for the learners. The link between the structure of the pattern (physical pattern) and the "formula" is lost. Unfortunately Zoe has removed the physical objects from the board when creating the flow diagram and cannot relate the formula to the flow diagram. Although Zoe exposes the learners to alternative illustrations of the given function, the movement amongst the representations (physical, diagram and rule) becomes a rote procedure and less about making sense of the function relationship.

The design of mathematically accurate explanations that are comprehensible and useful for learners as well as unpacking mathematical knowledge are recognised teaching tasks which teachers need to be able to do (Ball et al., 2008). However these are not easy for inexperienced preservice teachers and Zoe has tried to present and explain the concept of functions so it makes sense for learners. The shift from the flow diagram to the function rule is a big conceptual leap for the learners and she does not create the need for generality while the input values are

calculable. Perhaps if she had asked for the number of circles needed for Day 100, the learners may have been more inclined to look at the structure of the function instead of using recursion. Mason (2008: 86) suggests that “expressing generality” is not a strategy to be taught, used and tested but rather a holistic approach to “making sense mathematically”. He believes that children need to be encouraged and trusted to develop their own thinking and teachers need to use “scaffolding and fading” if they hope to be pedagogically effective (Mason et al., 2010).

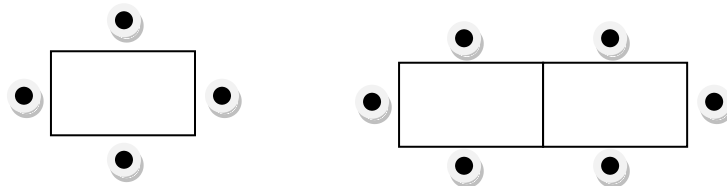
4.5.3. KATE - GRADE 5

a) Lesson outcomes

The lesson was titled ‘Early algebra – Tables and Chairs’ and the assessment standards included the following: perform mental calculations; use a range of strategies to check solutions; investigate and extend patterns looking for a relationship or rule; describe relationships in own words, determine input and output values; solve and complete number sentences and determine equivalence of different descriptions of the same relationship.

b) Lesson summary

Kate starts the lesson by outlining an imaginary real life problem which the children have to solve. The school is organising a dinner in the hall for forty people and need to decide how many tables would be needed. The tables and chairs (1 chair = 1 person) have to be arranged in a particular order which Kelsey demonstrates, with the help of the learners. Table 1 accommodates 4 chairs (people) and adding an additional table will seat 6 people.



She then records the data on a vertical table (t-chart) which is displayed on a whiteboard. Kate works with the learners and the physical objects until they have data for up to 4 tables and 10 people.

The learners are asked to describe what is happening on the right hand column and identify a common difference of +2.

Tables	People
1	4
2	6
3	8
4	10

The learners are then given a worksheet to complete in pairs, which has three columns and headings: dinner table, show picture and number of people. They are expected to complete the worksheet using the same pattern for starting with Tables 1 and going up to 7. Kate asks the class to identify the number of people needed for 10 tables and uses recursion to verify the answer. The class is then asked to identify the number of people needed for 50 tables and the teacher introduces an alternative method for working out the answer using a flow diagram (arrow diagram). The teacher uses the flow diagram to find the function rule using a guess and check strategy linked to the common difference. The learners are encouraged to look for a corresponding relationship between the input and output values. The lesson ends with a re-focus on the original problem of identifying the number of tables needed for 40 people and the problem is resolved using substitution. The children are given a consolidation worksheet to finish for homework.

The lesson involves a number of teaching tasks starting with the explanation of the mathematical idea i.e. activity of the tables and chairs needed for a dinner party. There is sequencing and demonstration of lesson content, unpacking the mathematical ideas to provide meaning for learners, recording of the input and output data, designing and explaining worksheets, using and managing different representations (physical objects, table diagram, flow diagram), understanding and interpreting learner responses, and asking productive questions. The lesson provides a rich source of teaching tasks and some aspects will be discussed in detail as they highlight particular demands placed on preservice teachers developing content knowledge for teaching.

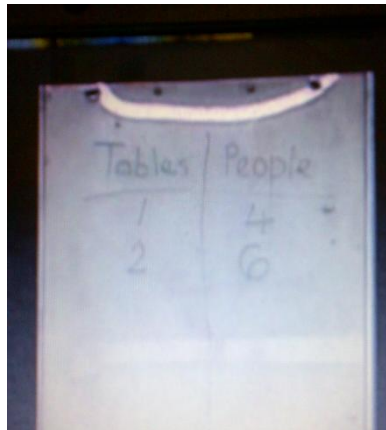
c) Specialised content knowledge*Selecting representations for particular purposes*

Kate can identify the mathematics content she is required to teach, based on the evidence in her lesson plan. She presents the mathematical ideas in a sequential and logical order. She starts the lesson selecting a representation which is linked to real life through the context of an imaginary dinner party. She uses physical objects: tables and chairs for the particular purpose of representing and solving the problem and to collect data for the t-chart/table. The mathematics is presented to emphasise logical relational understanding following careful and deliberate steps.

Teacher: The Grade 5s are planning a dinner party and we've decided that we want to have this dinner party in this very hall. So what we are going to make sure every time we think of something we are going to write it down. So how many grade 5 learners are there? How many in the class?

and

Teacher: Okay very nice, we can join the tables together and see how it works. So let's try and join a table and see how many people will fit now. Okay let's put the chairs around and we are going to count together and see. Okay so now let's see. Let's write that down so we can keep track of what we are doing. Okay so that is still not enough.



Tables	People
1	4
2	6

Figure 4.7: Preservice teacher's illustration of a function table

So let's try another table. Let's join one on here and count. I want you to wait and let's see. Okay, let's wait. Keep your hands up and wait, we will see now. Wait until everybody gets it. Let's wait and see. If you are not sure and you haven't seen it let's add it up and see. How many do we have? How many people can sit around the table?

Kate uses the tables as a way to check the validity of the student numerical guesses.

Teacher: Did you just guess? Okay. So that is why we have the tables here so that we can count and make sure. So if we have 3 tables. We worked out that there will be?

Learner: Eight people.

As the lesson develops, the learners are guided to approach the problem by looking at the output values only and identifying the difference +2 (recursive relationship). Unfortunately the constant difference is not related to the physical representations of the tables and chairs which limits the connection between the representations, the mathematical structure and limits the students' ability to identify the functional relationship.

Teacher: Okay, I like the way that you noticed that we are minus-ing. Grade 5s I want you to look here carefully and I want you to see what do we notice about these numbers over here (people column).

Tables	People
1	4
2	6
3	8
4	10

Figure 4.8: Preservice teacher's illustration of a function table

Learner: Counting in two's.

Teacher: Are we counting in 2's or what are we doing?

Learner: Times 2.

Teacher: Are we times-ing by 2? Think of your other operations. Grade 5's what are we doing to 4 to get to 6?

Learner: Divide.

Teacher: Divide okay I want you to think. We are getting bigger and what operations do we do when we get bigger.

- Learner: Plus.
Teacher: Plus, so what do we plus to 4 to get to 6.
Learner: Plus 2.
Teacher: Okay good, we are adding 2. So now I want you to do the same thing.
What are we doing to 6 to get to 8?

Although the students appear to be engaged in algebraic thinking such as studying change, predicting and justifying, they are not being helped to notice structure, problem solve or to generalise (Kieran, 2004).

Explanations that are comprehensible and useful for learners

Kate demonstrates an understanding of the development of functional thinking as the prediction of unknown states when using known data by challenging the learners to find the number of chairs needed for 50 tables (Blanton & Kaput, 2005b). She shows awareness of the pedagogical intent of the lesson and has planned accordingly, however she does not give the students the opportunity or time to discuss or problem solve and tries to scaffold their understanding through the introduction of a spider diagram. She wants the class to think about the co-variance of the numbers and tries to guide them to this through an alternative representation (Blanton, 2008). The class are not given the opportunity to discuss or use their own strategies. It would appear Kate has made some didactical decisions which she needs to follow to achieve the desired outcome of the function rule.

- Teacher: Okay so now if we want to work out say a big number like 50 tables are we going to want to add 2 every time.
Learner: No.
Teacher: So do you think there is another way we can maybe work it out?
Learner: Yes.
Teacher: Okay so we have this little machine that we call a spider diagram.

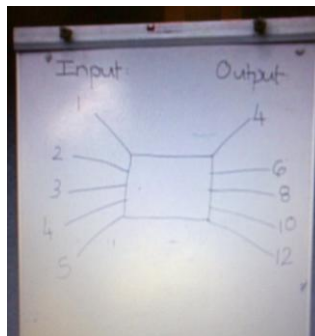


Figure 4.9 Preservice teacher's illustration of a spider diagram (flow diagram)

- Teacher: Have anyone of you seen something like this before?
- Learner: Yes.
- Teacher: Okay so what happens with a spider diagram? Okay so we are going to put something in the middle here. Okay so when we see a spider diagram we take a number which is called our input. Something happens in the middle and it comes out differently and we call this our output. So it's almost like if you're baking a loaf of bread. You put it in the oven like when it's dough, it bakes in the oven and then something happens to it, it changes and it comes up a risen loaf of bread. So now what we are going to need to work out is... What do we do to our numbers from our table to get to the number of people that we have over here? So we say our first one was 4, 6, 8, 10, 12. So now I want you to look at the different numbers and see what do we do to 1 to get to 4. You must times 1 by 2.

She tries hard to encourage the class to see the relationship between the inputs and outputs but they persevere with 'guess and test' (trial and error) method. She eventually decides to intervene and suggests a possible link between the difference and the function rule. Fortunately for her one of the learners makes a correct guess.

- Teacher: What that does it gives us a little clue. It teaches us a little clue that we are going to have to use that 2 somewhere.
- Learner: You put +2 in the middle.
- Teacher: Okay that tells us... 1×2 equals?
- Learner: 2
- Teacher: Then what do we do to 2 to get to 4.
- Learner: $\times 2$.
- Teacher: But remember we tried that one earlier. What is different? Gives us the same answer but different. We don't always have to multiply. We can also?
- Learner: Plus.
- Teacher: Plus okay good. Let's try that one. $1 \times 2 = 2 + 2 = 4$. Okay let's try the next one. $2 \times 2 = 4 + 2 = 6$. Is that right?
- Learner: Yes.
- Teacher: Okay now let's try another one. We want to make sure about what we say.
 $3 \times 2 = 6 + 2 = 8$.

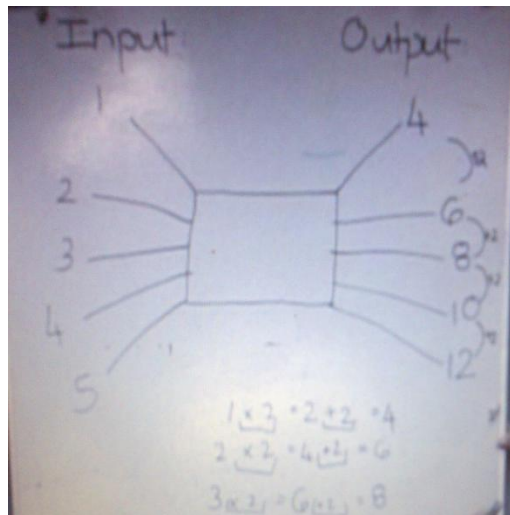


Figure 4.10: Preservice teacher's illustration of a flow diagram

Teacher: What do we see? What have we done? We worked out how we always get it. So now with this machine we have to write in here. Remember what we said earlier. If you're adding 2, you are writing the 2 in the middle. So what are we going to write in the middle here? $\times 2$ and $+2$.

The class has now found the 'magic formula' which can be used to calculate the number of people for any number of tables but Kate then changes the problem and instead of giving them another input value, she selects an output value i.e. doing and undoing (Mason et al., 2005). The learners are now expected to work inversely and luckily one of the children suggests 19 which happens to work. Unfortunately Kate does not ask how the number was found so the activity remains on the guess and test level and does not develop from discussion or argumentation of the mathematical structure of the problem or proof (Blanton & Kaput, 2005b).

Teacher: Okay so now we've just worked out the magic formula. If I said to you 3 500 tables we can say how many people will sit around it. So let's say remember we said we need 40 people that we need to sit. So now let's guess how many tables do you think we are going to need and we will try it we will test it out with our little formula we just worked out?

Learner: Nineteen.

Teacher: You think 19 tables. Let's try. 19×2 and what is that? 19×2 , yeah if you can't remember 19 and $19 = 38$. Remember you are forgetting to carry over the one. Okay so then we go $38 + 2 = 40$. Can we please give Lizette a round of applause for guessing that one? That was very impressive Lizette. Okay so now we've just worked out that if we

need to sit 40 people, we need to get how many tables in here. Remember the number.

Working with learners' ideas

Kate wants to engage her learners by encouraging them to think about how they are solving their problem. She expects them to justify their mathematical thinking and tries to build relational understanding through question and feedback.

Teacher: Okay so now I want you to think carefully. I don't want you to guess. I want you to think carefully. What do you think we would have? How many people will fit around if we had four tables? Think, think, think.

Learner: Ten.

Teacher: Would we have 10? Do you think she is right? Why don't you think she is right?

Learner: Because if its 8 people on 3 tables then you must count another 1 (table) to see if that one is right.

Teacher: Is that how you did it? So how did you get from 8 to 11?

Learner: Because in a table there sit four people but minus 1 then it's 11.

Teacher: Grade 5s I want you to look over here carefully (point to the people column in the table) and I want you to see what do we notice about these

Overall, it would appear Kate knows and understands the mathematics of patterns and functions for the Grade 5 level and is able to make use of the appropriate mathematical terminology. She uses growing patterns with real life objects to extend, describe and predict to eventually write a rule (in words or symbols) for the pattern. She is aware that most learners find it easier to start seeing the relationships within patterns by looking for difference (recursive patterns) but she tries to explicate the limitations of this process by asking for 50th step to move the class towards identifying a functional relationship i.e. generalise. She demonstrates and discusses the growth of the pattern, organises the information systematically and uses their analysis to develop generalisations about the mathematical relationships in the pattern (Van de Walle et al., 2007). She makes use of different representations: tables and chairs, drawings, t-charts/tables and flow diagrams to help support student thinking and to develop the functional relationship.

4.5.4. MIKE - GRADE 6

a) Lesson outcomes

This lesson is building on the concept of relationships between different things and the effect of change on each of the variables. It involves a similar context to Kate and is related to setting up tables and chairs for a party. The outcomes include defining relationships, investigating and extending patterns, looking for relationships and describing the function relationship.

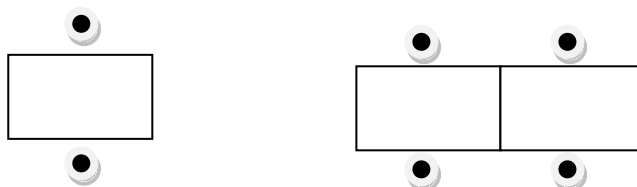
b) Lesson summary

The lesson detail and explanation is similar to Kate and has not been summarised a second time.

c) Specialised content knowledge

Explanations that are comprehensible and useful for learners

Although the table pattern problem is similar to Kate's pattern, Mike uses a different approach to the development of the function concept and forms connections between the tables and people in a different way. He starts by drawing one square table and puts faces to indicate people sitting at the table. He discusses the problem of seating people at a party after which he adds another table and some more people. He then has a general discussion about what is happening with the table and people involving the whole class



Teacher: Okay, our relationship is going to be between the people that I invite to my party and the amount of tables I have. Okay so look over here... On one side of our relationship we have people and on the other side of it we have?

Learner: Tables.

Teacher: Tables. Good. So how many people do we have at our one table?

Learner: Two.

Teacher: We have two people at our one table so how many tables are there?

Learner: One.

Teacher: There is only one table, hey! Alright and that's our relationship okay. For every one table how many people are we going to get?

Learner: Two.

Teacher: Two. What does this remind you of?

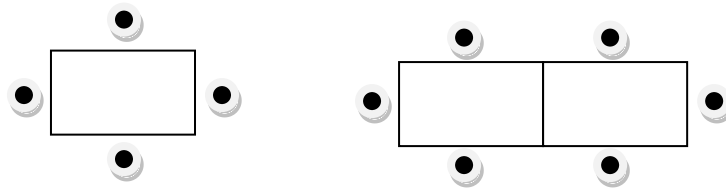
Learner: Ratio.

Teacher: There is a ratio going on there, hey!!!. 2:1 is our ratio. For every one table we are going to have two people. So who can guess how many people we are going to have at two tables?

Learner: Four.

Teacher: We are going to have four people, hey!

Having introduced the drawing and the ratio of the relationship between the table and the people, he then changes the original drawing to include two additional people at either end of the table. The data collected for the number of people and the total number of chairs is then represented in the form of a flow diagram to emphasis the relationship between the variables.



Teacher: Alright so now we have four people and two tables but now is there more space for more people here?

Learner: Yes.

Teacher: Where? I want you to come up and show me where we can fit more people. Please.

Learner: Here sir.

Teacher: Okay just stick a person there. Good I like it. Okay so now here we have two tables but how many people do we have?

Learner: Six.

Teacher: We have six people okay. So we have to change this. We gonna have to have six people over here. So just look at this through a flow diagram or a flow chart. Does anyone know what a flow chart is? Can you tell me what a flow chart is?

Learner: A flowchart is when you're linking one thing to another.

Teacher: Brilliant when you're linking one thing to another. Alright so what I'm going to do over here is I'm going to put a T what do you think that stands for. What do you think T stands for?

Learner: Tables.

Teacher: Tables and what I'm gonna do on this side is put a P for people. For people okay. Good and now this is going to represent our relationship about the tables. (Teacher follows with guidelines on how to draw the table)

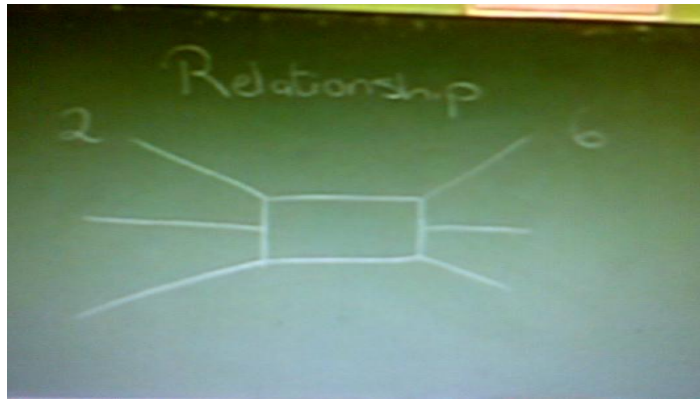


Figure 4.11: Preservice teacher's illustration of a flow diagram

Teacher: Alright someone tell me the amount of tables we have in this picture. Is how many?

Learner: Two.

Teacher: We have two and how many people do we have at the two tables. Let's give someone else a chance. I don't want shouting out the whole time. How many people do we have at the two tables?

Learner: Six, sir.

Teacher: Six, okay. Now how did we get from two tables to six people? We got two over here and there are four people like we said at those tables and then how many did we add?

Learner: Two.

Teacher: We added two hey. We added two. So what I'm going to do is I'm going to take this further alright. Remember that ratio we were talking about alright. I'm going to add another table. How many people do you think are gonna be at our three tables?

Learner: Six.

Mike then continues by emphasising the role of ratio and the additional two people needed each time to make up the total people needed for each table. He then asks the class to hypothesise the number of people that could be seated with 15 and 20 tables if the same seating arrangement is maintained. The learners are encouraged to discuss and draw solutions to the problems. There is no mention of magic box or recipes as evident in Kate's lesson and learners are encouraged to devise a plan and to come up with their own solutions. The mathematical terms: flow diagram and chart are discussed and explained, where necessary. Once the problems are solved, Mike brings the learners back to the number of people needed for 3 and 4 tables and helps them to describe the relationship in more detail.

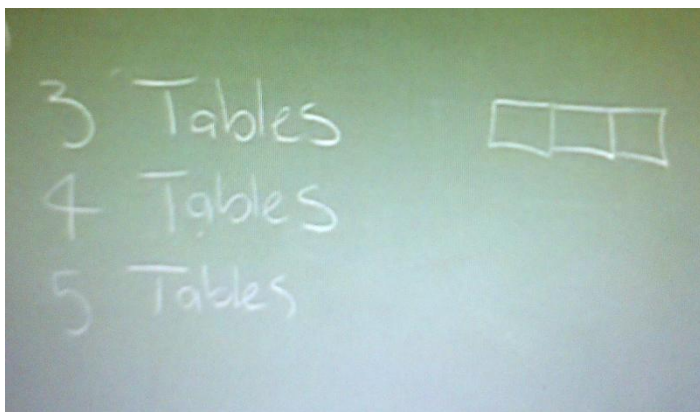


Figure 4.12: Preservice teacher's illustration of a pattern problem

Teacher: Okay for three tables, we had eight people. Quinton I want you to tell me... explain to me how you got 8 people. Just by looking at these 3 tables, how do we know that it is 8 people?

Learner: Three tables is six people, add 2 and you get 8.

Teacher: Just hold on. That is right so you said??

Learner: 3 equals to six people.

Teacher: What equals to 6? What do you have to do to the three?

Learner: Double.

Teacher: You have to double it or multiply by two. Two, hey, that is the same. Equals six people. Are we finished yet?

Learner: No.

Teacher: Why aren't we finished yet?

Learner: We need to add two people.

Teacher: We need to add those other two people hey plus two equals eight. Alright I want you to put up your hand if you understand it. This makes my heart happy hey. This makes my heart very happy. Alright, so here we go. I want someone to explain to me exactly the same way Franseen explained there with this next sum please. Let's try you boy. We have four tables. What do we have to do to get our people? Okay, so we are going to take our four tables and multiply it by two because there are two people per table and then what does that equal??

Learner: Eight.

Teacher: That equals eight. Are we done yet?

Learner: No sir.

Teacher: How many do you need to add?

Learner: Plus the two on the sides.

Teacher: We need to add our two on the sides hey. Who got ten? Did anyone get ten?

Learner: Yes, sir.
Teacher: Brilliant. 4 Tables
Learner: $4 \times 2 = 8 + 2 = 10$.

Mike chooses to work with the drawing of the table to help develop relational thinking between the tables and the people, it is only later that he moves to represent this information in the flow diagram as a way of helping learners to work with different input values e.g. 5; 17; 32; 149 tables. He tries to build from the special case towards a more general expression for any input value.

Working with learners' ideas

Mike decides later in the lesson that he wants learners to be able to work in the reverse order with input and output variables (doing and undoing - Mason) and poses the following question: If I have 68 learners, how many tables am I going to need? The first learner reply elicits an incorrect response and Mike is forced to work with the learner's response to correct the error.

Teacher: If I have 68 people how many tables am I going to need? I want to see more hands up. I want to see more people figuring it out. Cassidy.

Learner: 138.

Teacher: A hundred and thirty eight. Grade 6, I'm not going to ask if you are going to make noises like this. Cassidy, we have 138 tables for 68 people does that sound right?

Learner: That's wrong.

Teacher: Does that sound right? Do you want to try again? Should we have fewer tables or more tables? Let's ask someone we didn't ask before. Keshia.

Learner: 34.

Teacher: Thirty four. Who says that is right? Put up your hands if you say that is right? Is that right, 34? Who says it's another answer? Lee-Anne, how many tables?

Learner: 32 tables.

Teacher: Thirty two tables. Who says that is right?

Learner: I say it's wrong.

Teacher: Who has got a different answer? Guys what happened to putting up your hand. Keep it closed. Keeping it closed doesn't mean making noises in there as well.

Learner: 33 tables.

- Teacher: Thirty three tables. Who agrees with 33 tables? Yusuf alright please come and explain to me what you did to get to 33. Okay, I want you all to listen up. Nice and big okay.
- Learner: $33 \times 2 = 66 + 2 = 68$
- Teacher: You're just checking it is actually right hey but what I want you to do is I want you to walk me through to how you got to 33.
- Learner: $68 - 2 = 66$
- Teacher: So $68 - 2 = 66 \div 2 = 33$. Can anyone see the connection between this calculation and this calculation ($\times 2 + 2$)? Who can tell me the key phase that we did there? Who can tell me? Lisa.
- Learner: It's the opposite.

Mike handles the learner responses differently to Kate by trying to get the learners to explain their thinking and reasoning in solving the problem of finding the respective input value. While Kate accepts learners guesses and tests their answers without encouraging learner explanation and justification, Mike tries to help learners to make links between what they have seen in the drawing and their method of finding function values. While both methods result in the same answer, Mike provides a more useful way of thinking about functions as relationships rather than magic tricks and recipes. He builds on the physical and diagrammatic nature of the problem to create a more relational understanding of the function concept. Both pedagogical approaches highlight important aspects of teaching functional thinking and the challenges these pose for preservice teachers.

Blanton and Kaput (2005b) describes algebraic thinking as a process in which learners generalise mathematical ideas from a set of particular instances. They create generalisations through argumentation, and communicate them in age-appropriate ways. Although it would appear that Kate is endeavouring to do just this, she ignores the mathematical structure of the problem and limits the opportunity to generalise beyond trial and error. Mike seeks to establish links with structure and to build functional thinking and reasoning in a logical and developmental manner. If children are to develop algebraic thinking, they need to make use of symbols sensibly and meaningfully which means having access to various forms of representations and being able to move flexibly from one representation to another, as well as being able to operate on symbols meaningfully in context when necessary (Schoenfeld, 2008).

4.5.5. BRYN – GRADE 7

a) Lesson outcomes

The learning outcomes given by the preservice teacher for this lesson require learners to represent a pattern graphically, compare different forms of representations of the same functions, to develop further understanding and to build good attitudes towards their work.

b) Lesson summary

The learners have already completed some patterning lessons but have not yet linked pictorial images, tables and graphs. The assessment standards for this grade require learners to determine, analyse and interpret the equivalence of different descriptions of the same relationship or rule presented: verbally, in tables, by equations or expressions and draw graphs of a situation.

c) Specialised content knowledge

Explanations that are comprehensible and useful for learners

Bryn has previously developed the concept of function over a series of lessons. He is currently working with different pictorial diagrams to create patterns, to analyse and generalise for linear functions. This lesson starts with a square match pattern and involves the addition of three matches to create the next term in the sequence. The learners help the teacher to extend the pattern and relevant data is captured in a function table.



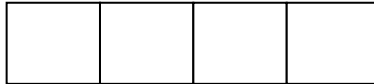
Figure 4.13: Preservice teacher's illustration of a square pictorial pattern

x	1	2	3	4	5	6
y	4	7	10			



Bryn's explanation involves learners in the extension of the drawing and the construction of knowledge. The development of the function concept is strongly scaffolded by the preservice teacher through the linkage of difference (+3) to the x3 times table. The learners appear to follow the prompts under the guidance of the teacher.

Teacher: So if I draw my next one onto this one. Where will I put it? Come show me. Okay so continue this pattern.



Okay so can you see we are adding the 3 all the time? So we say $3 + 3 + 3 + 3$. Where else do you see $3 + 3 + 3 + 3$ in maths? Where else do we normally see it? (pattern of 3)

Learner: Times table.

Teacher: What times table?

Learner: 3 times table

Teacher: Three times table. So it's 3; 6; 9; 12; 15; 18. So is it the same as this number there (pointing to the output value 4).

Learner: No sir.

Teacher: What is the difference? Number one is 4 so what are we doing.

Learner: We add 1

Teacher: Adding 1. Right so, it is $3 + 1$; $6 + 1$; $9 + 1$; $12 + 1$; $15 + 1$; $18 + 1$. Okay so if we look at this. How do you think we would write the formula for that. What were we doing first?

Learner: $x3 + 1$.

Teacher: Why?

Learner: Because it's $1 \times 3 + 1$ is 4.

Teacher: Okay and the next one.

Learner: $2 \times 3 + 1$ is 7

Teacher: Can you see that?

Learner: Yes sir.

Selecting representations for particular purposes

The preservice teacher then introduces another match pattern involving the formation of triangles and proceeds to explain the concept of difference in a similar fashion as previously outlined.



Figure 4.14: Preservice teacher's illustration of a triangular pictorial pattern

This activity involves the creation of a triangular match pattern, completion of a table and the search for difference. This is again linked to the times table but with the added inclusion of graph sketching. Bryn teaches the learners how to plot the x and y values from the table and highlights the difference (+2) aspect in the graph. The learners are presented with different representations of the linear function $y = 2x + 1$ and then given the opportunity to create their own graphs of the $y = 3x + 1$ function, which does not happen due to time constraints.

Teacher: What I want you to do is, first take one of these. So the line paper I gave you guys now, you get it with blocks right. So what I want you to do on the page, just look here quickly. I want you to draw my table that I drew there okay. So I want you to draw at least 20 blocks this way and 20 blocks that way. So y and x axis must have at least 20 blocks on each. Okay guys just stop there quickly. I don't think there will be enough time for everyone to draw their own one. I will just show you quickly on the board. Okay so what line is this? What points are these from? X equals and y equals. What is the equation for that line there, that points there?

Learner: $2x + 1$

Teacher: Remember that table there. Okay, so this table is $2x + 1$. It corresponds with this picture and the table corresponds with the co-ordinate's graph. This one with the co-ordinate graph okay. So let's do the one that we tried previously, the one that was... What was the previous one that we did? Y equals to?

Learner: $3x + 1$.

It is clear from the explanation and selection of representations that Bryn wants learners to understand that the function concept can be presented in different ways: pictures, tables and graphs. However there is little opportunity for learners to discuss what is happening in the transformation between the different representations or to understand the relevance of such information. Bryn tries to initiate such discussion towards the end of the lesson when analysing the graphs of the two linear functions but runs out of time to develop the content further.

Teacher: Okay so which one is adding 2. Okay so he (*a learner*) says that in this one here I'm adding 2 and in this one here I'm adding 3. So do you see a resemblance between the two?

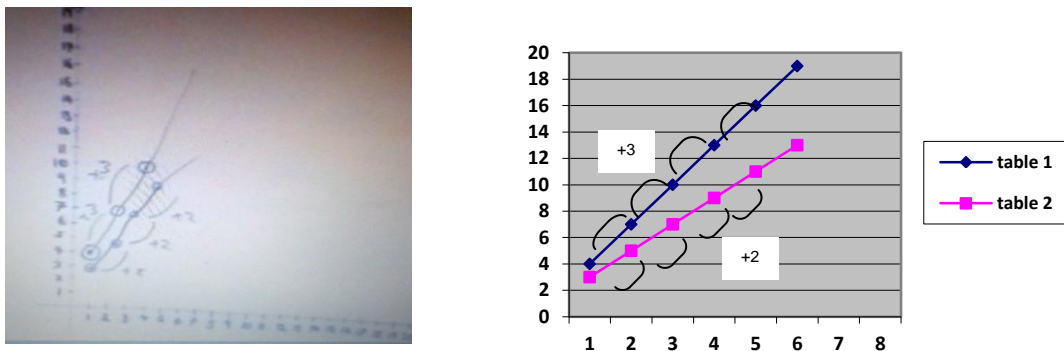


Figure 4.15: Preservice teacher's illustration of linear graphs

Teacher: So the more I add here, the steeper my graph is. So this number here will affect what is called the gradient of the equation, the gradient of the graph. So what I want you guys to do firstly, plot these two in two different colours for me.

This preservice teacher demonstrates an understanding of functional thinking which involves six activities that underlie functional thinking and the construction of functions (Smith, 2008):

1. engaging in some type of physical or conceptual activity;
2. identifying two or more quantities that vary in the course of this activity and focusing one's attention on the relationship between the two variables;
3. making a record of the corresponding values of these quantities, typically tabular, graphical or iconic;
4. identifying patterns in these records; coordinating the identified patterns with the actions involved in carrying out the activity; and

5. using this coordination to create a representation of the identified pattern in the relationship.

Bryn makes use of his knowledge of functions to help children “see and describe mathematical structures and relationships for which they can construct meaning” (Blanton, 2008: 6). He assists learners to generalise mathematical ideas from a set of instances and to express them in formal and age-appropriate ways but fails to establish those generalisations through the discourse of argumentation (Blanton & Kaput: 2005b). There is strong teacher guided instruction and prompting of answers which makes it difficult for learners to construct their own comprehensive understanding of the function concept.

According to Schmidt (2005: 528) curriculum coherence involves the logical sequencing of a topic to reflect the inherent structure of the discipline (mathematics) and in addition it also needs to focus on the interconnections between ideas within concepts. Bryn presents the pattern and function lesson in a logical and interconnected way through the mapping of the linear functions across different representations with a focus on arithmetic (times table) as structure rather than on arithmetic as product (computation) (Warren, Mollison & Oestrich 2009). However there is little point in being able to complete sets of instructions and tasks if learners are not making effective and purposeful use of symbols in ways that are inherently sensible and meaningful (Schoenfeld, 2008). Understanding functions has to involve ways of thinking within activities such as analysing relationships between quantities, noticing structures, studying change, generalising, but there also needs to be opportunity for learners to problem solve, model, justify, prove and predict (Kieran, 2004).

4.5.6. Summary of preservice teachers’ SCK and teaching

The mathematical work of teaching requires teachers to master a number of tasks such as giving mathematical definitions and explanations, selecting and using appropriate representations, working with students’ ideas, making judgements about the mathematical quality of instructional tasks and asking productive questions (Ball et al., 2004). Developing knowledge to teaching early algebra requires preservice teachers to have common content knowledge as well as specialised content knowledge. This knowledge is then transformed and used in the classroom setting in various ways and in different contexts. There is no doubt that the social,

economic and political setting of the classroom impacts on the pedagogical decisions and actions in the classroom and it is particularly difficult for the preservice teacher to feel completely comfortable and confident in another teacher's classroom. Given the constraints of the classroom, the analysis of these early algebra lessons is important in helping to understand what knowledge preservice teachers have of early algebra and its impact on their teaching. This analysis looks at the actions of the teachers and learners and does not make judgements about the overall teaching ability of the preservice teacher.

The preservice teachers' lessons selected for analysis made use of physical and pictorial representations as well as real life contexts to create and extend patterns. They focused on the development of functional thinking through a series of activities and encouraged learners to actively engage in the lesson. There was an emphasis on establishing relationships between variables in order to determine input and/or output values in a variety of ways using verbal descriptions; flow diagrams; and tables (DoE, 2002). The patterns were all linear functions and there was no mention or discussion of alternative types of pattern or different functions. There were some examples of links between the structure and generalisation of patterns as well as developing interconnectivity between different representations of functions such as drawing, tables, flow diagrams and graphs (Mike and Bryn). Learners were also given opportunity to work with input and output values both in doing and undoing patterns. These preservice teachers demonstrated specialised content knowledge for teaching early algebra but they also highlighted difficulties/demands which they faced in the task of teaching.

There are particular tasks which appeared to challenge this group of preservice teachers in teaching early algebra across a variety of grades in the primary classroom. They did not always avail of the opportunity to link the physical objects and the inherent pattern structure to help in the generalisation of the pattern. The concrete and pictorial tasks helped learners to create and extend patterns which were tabulated or represented in flow diagrams, followed by analysis of the pattern. The analysis was often restricted to a focus on common difference through the recursive approach resulting in 'guess and check' to find a formula to generalise the pattern. Some of the preservice teachers found it difficult to move learners from recursive thinking to co-variance and correspondence thinking as a result of the choice of data representation. Using both tables and flow diagrams, in the same

lesson, to introduce the concept of function seem to cause problems for both learners and teachers.

The preservice teachers were sometimes forced to resort to teaching generalisation as a strategy instead of a way to understand and describe relationships between numbers. They made use of scaffolding to develop functional thinking but sometimes without success and had to resort to giving the function rule. Learners were encouraged to guess and check various options to describe the pattern without enough time to think, discuss and argue various solutions. There was little or no time for problem solving and predictions were strongly guided by the teacher. The preservice teachers' questions focused primarily on prompting responses and provided little opportunity to probe and develop understanding of pattern as function.

The third level of analysis related to the Maths 2 course and the teaching practicum and looked at the focus group interviews responses for illustrations of preservice teachers' development of knowledge for teaching early algebra. The findings are described and interpreted in the next section.

4.6. The Maths 2 course

The preservice teachers were given the opportunity to reflect on the year long Math 2 course in the form of focus group interviews using four topic headings: content (disciplinary mathematics knowledge), theory (academic text read and discussed in the course), methodology (teaching strategies and classroom management) and teaching practicum. The purpose of the focus group interview was to elicit their thoughts and opinions on the course, to inform future mathematics education courses and to determine links to the development of knowledge for teaching early algebra. Three activities were identified from the verbal responses as important in the design of the Maths 2 course: current literature on early algebra and children's thinking; learning community and teaching practicum. Each of the categories is explained and extracts from the interviews have been included to support the explanations.

4.6.1. Current literature on early algebra and children's thinking

The design of the Maths 2 course included specialised content knowledge of early algebra, in the form of book chapters and journal articles related to the teaching and learning of EA. The preservice teachers felt the articles helped them understand

where learners were coming from and the reasons why learners think in different ways. They discovered that it was not only important to recognise the correct answer but to also understand the thinking behind it.

The articles help you understand the learner and where they're coming from and sort of better (improves) your teaching in order to better (improve) their knowledge and their understanding. So you need to know what they understand and work on what they don't understand in order to help them. (FGI)

The articles always show the different understandings of the learners..... I saw this is why the child did this, why did they do it because they seeing it in a different way so I have to explain what they saw but if it's incorrect I have to show them but if you do this then you'll find it and this is what this article showed. You don't only look at the correct answer but the thinking behind that answer and that is what the articles brought us. (FGI)

In terms of teaching the maths you can actually see how the articles have helped us to realise how to link things up and while we teach maths, what we teach in one day and not just show the kids theory (content) but to show them why it is like this and get them to explore, so that when we are maths teachers one day we would be equipped to get them to think algebraically. (FGI)

Some of the preservice teachers found the articles helped them to understand mathematics teaching and learning better through the written explanations and examples supplied and the follow up class discussions. There were opportunities to read, discuss, make connections and see relationships which helped build their confidence. This was particularly relevant to the last article which traced the development of algebraic thinking from primary school right through to calculus. It helped to consolidate much of what the preservice teachers had learned during the course and the teaching practice experience.

.....they (the journal articles) helped me to understand the mathematics far better because it gave a written explanation on what we were actually trying to do...(FGI)

So that's how I feel after the readings that I can use the knowledge and I can apply it and obviously have a better understanding. (FGI)

....talking about the kind of maths that we are learning and the methods we are using to teach children these days as opposed to the old days. And it will be

some person will say well there is only one way to teach math and you can actually say well do you know that in this place they've tried this and do you want to read this article. (FGI)

There is no consensus on which approach to use when designing courses on developing teachers' knowledge for teaching; however it would appear from the preservice teachers' perspective that reading and discussing of current literature and children's thinking on early algebra is helpful in the development of knowledge for teaching. It provides opportunity to engage with learner's thinking and to make sense of learners' problems and solutions. It acts as a validation for some when challenged to justify the inclusion of early algebra in the younger grades.

4.6.2. Learning community

According to Wenger (1996, 2006) a community of practice is a "group of people who share a concern or a passion for something they do and learn how to do it better as they interact regularly". The preservice teachers in the Maths 2 course have started the process of initiation into a community of preservice mathematics teachers through participation in solving mathematical problems in a supportive and safe environment. Borko et al. (2005b) suggests that participation in a learning community is a crucial element in meaningful learning and involves rich tasks, opportunity for student explanations and a shared process of learning.

One of the strategies used in the Maths 2 course was to actively encourage preservice teachers to work together and across different phases: Foundation and Intermediate/Senior. The course was targeted at preservice teachers to help build an understanding of how early algebra is related over the span of mathematics included in the curriculum, and across grade levels (Bair & Rich, 2012). The feedback from the preservice teachers highlighted the importance of making time to get to know one another. This helped with their motivation and enjoyment through the support and the sharing of ideas. They felt able to take risks and to challenge one another without feeling threatened and felt comfortable to speak about mathematical problems and to problem solve difficult questions.

It was a good thing because then we got used to the other part of the class and everyone else in the class because then we can speak to the rest of them even though we had our own groups but we could still be comfortable enough to speak, even to ask a question so that we could be comfortable with our class. (FGI)

Our group also, the way we thought and we were like no this is how we did it and that is how you did it. We spoke a lot and Z would catch me in the middle of the passage and I would like what now and she would say I have a question. I was doing Maths last night. Did you figure this out and I'm like, oh, not yet. So we would like sit with it and figure it out. (FGI)

So it's almost like you can take the risk here and everybody in our class, I don't know if we just had a very nice group.....and people were actually trying to figure out and see now why did she (teacher educator) do that, okay well see like how she figured it out and whatever and then they tried to explain to you this is where you went wrong. I figured it out this is where you went wrong. (FGI)

The preservice teachers recognise the value of creating a mathematics environment which encourages risk-free participation. Through co-operative group work, they learn to encourage and support one another in the task of developing knowledge for teaching early algebra, both CCK and SCK. The verbal responses of the preservice teachers expound the benefits of working within a community of practice both in their attitudes and commitment to learning which ultimately shapes their identity as a mathematics teacher.

4.6.3. Teaching practicum

There is a tendency in many traditional mathematics education programs to separate content, pedagogy (mathematics didactics/methods) and teaching practicum. The subject matter knowledge is often addressed in the content course and delivered by mathematicians, while the mathematics methods course is used to study how children learn mathematical concepts and skills and to how to teach particular mathematics ideas to learners (Strawhecker, 2005: 2). The location of the teaching practicum slot often depends on the design of the program. The Maths 2 course made a purposeful connection between knowledge for teaching early algebra and teaching practicum to give the preservice teachers an opportunity to apply what they had learned about early algebra in the classroom. The teaching practice experience provided an opportunity for authentic learning in the real classroom context which has been shown to make positive gains in pedagogical content knowledge (Lowery, 2002, in Strawhecker, 2005).

Yeah, by going through the whole lesson I got to see how they think things through and where they struggled because I wouldn't have been able to know that that was what they would struggle with and by seeing what they were

struggling with, the next time I teach it I can be prepared for that. Okay, I know they're going to struggle with this, so let me first cover that side with it and let's keep track of our findings like this so while they doing it they don't end up struggling with that. (FGI)

I think with Teaching Practice we were able to use what we've learned in the class and be able to actually put it into practice in an actual classroom and to see physical responses from the kids and how they answer questions compared to just seeing the examples and seeing the results from other studies done and now to be able to see it first for your own class. (FGI)

The preservice teachers' comments illustrate the important impact of the teaching practice experience on the development of their knowledge for teaching early algebra. Through the teaching experience, preservice teachers became aware of many of the important pedagogical issues related to the teaching and learning of early algebra: expectations of learners, questioning skills, planning and design of the lesson, giving explanations, choosing examples and representations as well as working with learners' ideas. They learned to modify and adjust their knowledge and understanding of mathematics teaching through the deliberate link between content, theory and practice. Strawhecker (2005: 11) findings suggest the "impact of mathematics field experience (teaching practicum) combined with a methods/content, perhaps, shows the greatest promise in preparing primary teachers to teach mathematics". Her research is influenced by the work of Ball and colleagues and suggests the need for a model which integrates content knowledge and pedagogy in the context of teaching. However, she acknowledges the lack of available models to effectively prepare preservice teachers to teach mathematics at this stage.

While there were elements of the teaching practicum that were useful in the development of knowledge, there were also some issues which arose and were identified as problematic and restrictive in the process. Firstly, the early algebra lesson was designed in isolation from other mathematics lessons and often did not fit with the current topic of the classroom. This was due to the fact that we did not want to impact too much on the teaching time and could not ask for additional lessons. It meant that learners got a brief exposure to early algebra and not to other early algebra activities. The preservice teachers could not follow up on things they had recognised as problematic and had to rush to include everything in the lesson.

Secondly, learners were often expected to work with concrete, pictorial or real life data, to represent the pattern data in a table or flow diagram, look for relationships and generalise the pattern, all in the same lesson. This could also have been part of the reason that some of the lessons felt formulaic and contrived for the preservice teachers and learners.

Thirdly, it was difficult for preservice teachers to select a sequence of early algebra activities when they knew they would only have the one lesson to introduce and develop the concepts and skills. In saying that, it did give the preservice teachers the chance to experience an operational (process) approach to the teaching of early algebra instead of a structural (object) perspective only (Carraher & Schliemann, 2007). They had a chance to apply in practice the content and theory they had learned in the Maths 2 course and see learners enjoy and understand the lesson which surprised some of the preservice teachers.

I was unbelievably surprised at how quickly my maths class that I knew wasn't good at maths got the algebra lesson that we did in front of the video. How quickly they managed to get the concepts. I re-watched my video and I realised that I went back over things that they'd already got. I was just wasting time because I was in my head still thinking they haven't got this, yet they actually had got it. (FGI)

The illustrations from the focus group interviews highlight the role of the Maths 2 course and teaching practicum in building an “awareness of the conceptual understanding “ of early algebra and helping preservice teachers to develop a fundamental understanding of mathematics that is “deep, broad and thorough” (Ma, 1999: 124). Preservice teachers learn more about learners and mathematics through engagement with content and theory (CCK and SCK) and the practical experience of teaching early algebra. This combined experience highlights the complexity of teaching and learning mathematics and makes preservice teachers realise teaching mathematics involves both knowledge and understanding of the necessary concepts and processes and pedagogy. There is a need to be familiar with learners and their mathematical thinking and to have a mathematical knowledge of the design of instruction of an early algebra lesson (Shulman, 1986; Ball et al., 2008). Providing content and theory without some opportunity to experiment in practice would have given a limited conception of what it means to

teach early algebra, hence the inclusion of the practicum component into the course to develop systemic enquiry into the teaching and learning of early algebra.

4.6.4. Key learning principles for the Maths 2 course

There were some key learning principles which emerged from the Maths 2 course that preservice teachers felt guided their development of knowledge for teaching early algebra. These principles resonated with some of Kilpatrick et al. (2001) strands of mathematical proficiency: conceptual understanding; strategic competence and adaptive reasoning.

a) Conceptual understanding

The course content and methodology was a new experience for many of the preservice teachers. Their school practice of learning mathematics was identified as a fragmented set of topics that had to be ‘learned’ and ‘remembered’. They felt they did not have to understand what they had learned but rather to be able to recognise and use the mathematics appropriately. Through their participation in Maths 2 course they identified the importance of understanding mathematics as well as being able to do the mathematics which helped with overall retention.

It’s important to teach them (learners) the “Why” and “How” you got to, so that they can understand the concept and they can remember it.(FGI)

Because we (preservice teachers), need to understand what we do. We make sense of it that way. So that you can see there are many different like answers just for one thing and then one answer you may find actually easier to do, the one might be easier than the one that you made rote learn into us. Somebody else’s method might be this, okay I understand your way. I’m going to use your way and somebody else might say I prefer this way because it’s a lot more understandable... (FGI)

Conceptual understanding is one of the five strands of mathematical proficiency and “refers to an integrated and functional grasp of mathematical ideas” (Kilpatrick et al, 2001: 118). Teachers and learners with conceptual understanding know more than mere facts and procedures. They understand why a mathematical idea is important and how it can be used which enables them to integrate new ideas through connection to what they already know. It also helps to provide a “foundation for remembering facts and methods”, “solving new and unfamiliar problems” and

“generating new knowledge” (Kilpatrick et al., 2001: 11). It is evident from the preservice teachers’ comments that they have developed an understanding of the need to move beyond a procedural understanding of algebra (how) to an awareness of ‘why’ it is so.

b) Strategic Competence

Another strand of mathematical proficiency is “strategic competence” which is the ability to formulate problems mathematically and to devise strategies for solving them using concepts and procedures appropriately (Kilpatrick et al., 2001: 124). There were many comments related to solving algebraic problems. During the course, preservice teachers were given problems to solve in class and others to take home. Class work involved working in groups and sharing ideas, strategies and solutions. They were encouraged to try different methods for solving the problems and to share and justify their findings. Through these activities, they began to engage with problem solving and to recognise the value of this mental process. They did not need a particular method to solve a problem but preferred to trial and error various options and developed an attitude that ‘you can go to it from any angle and figure it out and do it backwards’. Solving problems in different ways helped the preservice teachers to understand the problem better and to recognise that thinking in this way helps learners to understand what they are doing. When they struggled and made sense of algebraic problems they learned to acknowledge the importance of having a deep and profound understanding of the content knowledge that needs to be taught (Ma, 1999).

Don’t just look at it from one way, a problem has a whole lot of entrances, you can tackle a problem from anywhere and its things like that so when I teach one day I want to explain the “why and the how” but it’s not something they have to learn off by heart and that’s I think..... it’s kind of something that you did that really helped. (FGI)

Yeah, because you can figure it out because what you know and more importantly what you understand. You basically use your own logic to get to your own conclusion to get to the same thing. (FGI)

Also in high school your motivation is just to get a good mark and to pass whereas now our motivation is to know and understand what we’re learning so that we teach it because if we don’t get it, sure we might fail the year but we also won’t be able to teach it to the kids. It’s intrinsic. (FGI)

I found in school the content was very isolated whilst here there are lots of connections and like oh my word ah ha moments. That was fun for me as well. It's like if I can't do it this way I'm going to try my own way and it's gonna work and I'm gonna get the same answer. (FGI)

Like K... said in high school if I had gotten that question I would never ever in a million years have thought to think like okay I have this that I know I need but I also know I need this used in an equation must give me that answer. So let me try and use the answer to try and get what I actually should have had at the beginning. I would never have thought about it like that. (FGI)

Problem solving is a crucial part of mathematics learning. Learners need to be able to understand a problem, devise a plan, carry out the plan and evaluate the solution (Polya, 1957). It is evident from the preservice teachers' comments the algebra tasks they did in class gave them the opportunity to problem solve and to develop their metacognitive skills. They devised plans using various strategies such as guess and check, looking for patterns, drawing pictures of problems, working backwards, experimenting, and became problem solvers themselves.

While the content of the course highlighted what teachers need to know about algebra, it also focused attention on how a teacher comes to know in the development of knowledge for teaching. Fuson et al. (2005: 236) recognise that "learning about oneself as a learner, thinker and problem solver is an important aspect of metacognition". They identify three elements of instruction that support metacognition: an emphasis on debugging (identify, understand and correct errors); internal and external dialogue (reflect on and describe mathematical learning); and seeking and giving help (engage and solve problems but seek and give help when necessary). It is evident from the preservice teachers' comments that these elements were part of the Maths 2 course and helped them to become more aware as mathematics teachers. The preservice teachers were given the opportunity to "embody modes of mathematical enquiry themselves" through becoming mathematicians and not just to model the behaviour they wish learners to have but to make a classroom "a place in which mathematics is performed" (Watson, 2008 in Burton, 2009: 5).

c) Adaptive Reasoning

Much of the conceptual understanding developed through making connections within the mathematical content of the course. The course gave preservice teachers the chance to learn mathematics through looking for relationships, recognising and accepting alternative strategies and solutions and identifying linkages within the topic of algebra and early algebra. They felt challenged to think algebraically through making conjectures, having discussions, generalising and being forced to justify their decisions.

You (teacher educator) are challenging us (preservice teachers) more to make our own conjectures and to generalise ourselves so you are not standing in front of the board and just saying this is that and we are just listening and watching. You are not telling us, you have to discuss, you have to think, you have to make your own conjectures and you have to do it yourself. It's what do you know, what do you see. Its more challenging and you learn more like that. (FGI)

I mean we did that like rote learning (school experience) but the thing that stuck out for me this year with this content was the connections between algebra and graphs, patterns and algebra and then linking from patterns to graphs essentially as well. Because I definitely had the most fun and was the most challenged by the beginning, with finding the patterns and then finding, wow, this is a $mx + c$ graph and you can actually put this pattern onto a graph. And because school does algebra in graphs you never made that connection and I think that was the most interesting thing, is trying to find out how patterns go into algebra which go into graphs, like everything is related to each other. (FGI)

The practice of teaching mathematics in separate distinct topics and as unrelated pieces of knowledge does not help to develop learners who can think algebraically. It is important to develop algebraic thinking through activities which encourage learners to experiment, argue, make links, conjecture, justify, generalise and prove. It involves adaptive reasoning i.e. "Using logic to explain and justify a solution to a problem or to extend from something known to something not yet known" (Kilpatrick et al., 2001: 9). Teachers need to be able to provide mathematical explanations, respond to learners' ideas and make connections between mathematical actions, symbols and pictorial representation while teaching (Askew et al., 1997).

4.7. Conclusion

The focus of this study was to understand how knowledge for teaching early algebra develops as a result of an early algebra course and teaching practicum. It investigated the development of knowledge for teaching early algebra in terms of both common content knowledge and specialised content knowledge for teaching early algebra as espoused and enacted by preservice teachers. The first part of the analysis in this chapter looked at how preservice teachers verbalised and wrote about their experiences. This was followed by a detailed look at the actions of some of the preservice teachers when teaching an early algebra lesson. The final part of the analysis related to the Maths 2 course and teaching practicum and their experiences of the course to help identify what aspects of the course had helped in the development of knowledge for teaching early algebra.

The understanding of early algebra demonstrated in the verbal and written responses of the preservice teachers was understood from a number of different perspectives informed by the literature and activities within the Maths 2 course and teaching practicum. The preservice teachers' responses and actions illustrated different understandings of common content knowledge and specialised content knowledge needed for teaching early algebra. They were able to reflect on the successes and challenges involved in teaching an early algebra lesson and highlighted pertinent issues related to the knowledge needed to allow teachers to engage in teaching tasks. There were particular challenges in the transformation of knowledge for teaching involving explanations, using representations and working with learner responses.

The Ball model provided a useful lens in the elaboration of knowledge for teaching early algebra. However the findings of this research highlighted the tacit nature of knowledge needed for teaching and the added complexity of defining and capturing this knowledge (Rollnick & Mavhunga, 2013). Specialised content knowledge is the knowledge needed for teaching and does not always exist separately from common content knowledge of mathematics as illustrated in many of the responses. It is knowledge used in practice, it is implied and not easily verbalised by teachers or preservice teachers. Equally common content knowledge has its own complexities and appears to involve knowledge of mathematics as well as school mathematics content not easily recognisable as content. The inter-connected nature of common and specialised content knowledge presents a challenge for those involved in the

defining of knowledge for teaching but also highlights the crucial need to clarify the features of content knowledge for teaching that most influence practice.

The development of knowledge for teaching early algebra is a complex and multi-faceted concern which requires detailed and ongoing investigation. The work of mathematics teacher education is imperative in helping clarify the complexities involved in developing knowledge for teaching. This research helps capture some of the issues for preservice teachers both in terms of knowledge development and early algebra, it highlights important tasks of teaching which are problematic and helps give a better understanding of common content knowledge and specialised content knowledge for teaching early algebra.

CHAPTER 5. DISCUSSION AND CONCLUSION

5.1. Introduction

The purpose of this study was to understand the development of preservice teachers' knowledge for teaching early algebra as a result of the Maths 2 course and teaching practicum. The study attended specifically to the development of common content knowledge and specialised content knowledge for teaching early algebra and to the contributing aspects of the Maths 2 course and teaching practicum to the development of knowledge for teaching. Data was collected through the analysis of video recordings of their early algebra lessons and the ascribed meanings that they gave to their experiences in the form of lesson reflections, video questionnaires and focus group interviews. The research was guided by the following research question and sub-questions:

Research question:

How does preservice teachers' knowledge for teaching early algebra develop as a result of an early algebra course and teaching practicum?

Research sub-questions:

1. What manifestations of knowledge for teaching early algebra do preservice teachers illustrate in their verbal and written feedback taken from an early algebra course and teaching practicum?
2. What manifestations of knowledge for teaching early algebra do preservice teachers demonstrate in their teaching of an early algebra lesson during teaching practicum?
3. What aspects of the early algebra course and teaching practicum contribute to the development of knowledge for teaching early algebra?

This chapter continues with a summary of the findings from chapter 4 in section 5.2 which is followed by a discussion of key aspects of the findings (5.3). There is a presentation of the implications and recommendations from the study in terms of teacher knowledge; early algebra and professional development of preservice teachers (5.4). The chapter concludes with the role of teacher education in the development of knowledge for teaching (5.5).

5.2. Summary of findings

The summary that follows is a brief outline of the findings presented in chapter 4 for the purpose of situating the discussion that follows. The findings were organised into four sections and tried to understand how knowledge for teaching early algebra developed by looking at what knowledge emerged as a result of the Maths 2 course and teaching practicum, including factors that emerged as contributors to this development:

- Common content knowledge of early algebra
- Specialised content knowledge of early algebra
- Specialised content knowledge in teaching an early algebra lesson – video lesson examples
- The Maths 2 course

5.2.1. Common content knowledge of early algebra

The findings for the verbal and written responses indicated that preservice teachers' understanding of early algebra (EA) could be categorised in five ways: EA as generalisation; EA as a process of functional thinking; EA as modelling; EA as global meta-level activity and EA as seeing arithmetic operations as instances of functions.

This means that early algebra did not take on a specific meaning for preservice teachers but was rather a representation of different aspects of early algebra. For some, early algebra was a special opportunity to develop a particular kind of generality in learners' thinking (Lins & Kaput, 2004; Mason, 2008), for others it was connected to the activities that underlie functional thinking (Smith, 2008).

The focus of my lesson was identifying the common difference and recognising the pattern, and recording information in a table through the matches' activity (square) learners had to see, observe and do. Maybe if learners had to first create the 4th, 5th and 6th square with the matches on their own and see if they can identify and pick up a pattern (add 3) maybe this would have been better.
(S10 - LR)

Early algebra was also about understanding relationships between variables in real-world contexts (Sanchez & Llinares, 2003) and included ways of thinking within

activities such as analysing, studying change, noticing structure, conjecture, justification and proof (Kieran, 2004).

My goal upon approaching this school was to show them that algebra is real and not abstract and this is the direction that I followed in my Mathematics lesson - to put Mathematics into context. (S8 - LR)

Early algebra involved seeing arithmetic operation as functions through thinking about relations among sets of numbers and emphasised the algebraic character of arithmetic (Schliemann et al., 2007).

I liked the fact that it was possible to link the concepts of doubling and halving to other mathematical concepts like patterns and algebraic thinking. (S15 - LR)

The findings in this section (5.2.1) revealed an awareness of the challenges of making early algebra accessible and relevant to learners through the development of functional thinking and the use of pattern activities. The preservice teachers experienced problems in helping learners to move from recursive patterning to identifying co-variant and corresponding relationships and in using questions to support learners' attempts to generalise. There were also issues in the purpose and use of models (representations) to teach functional thinking and the approach to functional thinking using a set of activities. The preservice teachers revealed a shift in thinking about algebra as procedures and rules towards algebra as a process of developing functional thinking through the investigation of patterns and relationships.

5.2.2. Specialised content knowledge of early algebra

The preservice teachers specialised content knowledge for teaching focused on six tasks of teaching: defining, explanations, representations, working with learners' ideas, restructuring tasks and questioning (Kazima et al., 2008). The written and verbal responses revealed that the explanations were often teacher directed and did not necessarily give priority to mathematical meaning or justification of the steps involved or the ideas used. The preservice teachers made use of real-life contexts to help make the explanations more real and accessible for learners. They used language in explanations either in a technical way or to express mathematical ideas but they did not talk explicitly about the meaning and use of the mathematical language of functions.

For the conclusion, I tried to get to see if the learners were able to find the rule, but they were not able to, so I guided them to the right answer, which was not the absolute correct thing to do but they all did seem interested even though they were not thinking themselves. (S13 - LR)

The knowledge of representations found in the preservice teacher responses revealed knowledge of a variety of different forms of representations: physical objects, pictures and diagrams, real life examples and symbols to develop the concept of a function which were age appropriate for the learners.

I felt having the real life examples of the tables helped the learners see what we were doing, and maybe have an idea of where we were trying to go with the maths. It was not an abstract thing to them, but something they could see and touch. (S25 - LR)

They tried to build connections between the different representations but did not always make the links mathematically explicit eg. discussing the connection between verbal and written generalisations. The representations became tools (products) in the development of the functional thinking and opportunities were lost to use the models as cognitive roots to help highlight the processes involved in functional thinking.

I chose to use a table to represent the information because I thought it would be easier for the learners to identify a pattern and the function rules. However, I think that perhaps the spider diagram (flow diagram) may have been better suited for this purpose. (S23 - VQ)

However, I noticed that the learners got confused when I showed them how the data in a flow diagram can be put into a table format. This was a new mathematics concept I wanted them to learn, however I think that majority of the class were not really ready for it. I should do more lessons using tables with them for future lessons to better this skill with them (S22 - LR).

The task of working with learners' ideas involved a number of different tasks including interpreting and making mathematical and pedagogical judgements about learners' questions, solutions, problems and insights; responding productively to learners' questions; and assessing learners' mathematical learning and taking the

next step. The preservice teachers' responses revealed these tasks proved to be challenging to manage but having taught the lesson, they were more aware of the issues and better able to address them in future lessons. They realised that their intention to involve learners in the construction of knowledge was restricted by their perceived need to follow the planned lesson and they did not know when to stop working with learner contributions and move to scaffolding learning.

My biggest problem with the lesson was my ability to understand and work with the learner's answers (when they weren't what I was hoping for). I would get stuck and nervous that I was losing the other learners in the class and I was afraid to drift too far away from my planned lesson. I think in the future I need to be more flexible in case my learners don't always think in the same way as I do. (S25 – LR)

This is where I faulted. I gave too many learners the opportunity to pursue their contributions and it was irrelevant as they were all giving the same methods. I thought that I was going to get different explanations from the learners, but I did not. (S9 - VQ)

Knowledge of restructuring tasks requires teachers to be able to plan and adapt activities which are appropriate for learners and enable them to work productively (Ball et al., 2008). The responses from the preservice teachers indicated awareness of activities needed to develop functional thinking: engaging in a problematic within a functional situation; creating a record and seeking patterns and mathematical certainty (Smith, 2008: 143). They were able to identify some of the difficulties in task design such as differentiating activities for different learners and connecting activities in a meaningful way. They learned through the reflective process of design, implementation and modification of tasks about the purpose of activities for teaching and about the ambiguities they sometimes transmit to learners (Zaslavsky, 2009).

Next time I should think of activities and how they connect and lead up to next activity so learners can understand better. Activities should have been more basic for the type of class I had, I should have done triangular match activity then square matchstick activity (S10 - LR)

I limited the lesson somewhat, in the sense that I did not plan for much extension with the work I was teaching and so I would recommend that various options

should be kept on hand just in case you need that little bit more, or less depending on the situation. (S26 - LR)

The final task for teaching: questioning was particularly difficult for preservice teachers and mentioned a number of times in their responses. They recognised the weaknesses in their questioning ability in terms of the levels and types of questions needed to build conceptual understanding of functions. They were prone to prompt learner thinking and struggled to ask questions which were fluid and coherent. There was an awareness of the need for better planning of questions and for different levels of questions, but also recognising that questioning skills develop over time and with experience in the classroom.

My questions could have been clearer, should have had more questions beforehand. I guided the learners too much. I didn't give them enough room to think on their own, to explore with maths. (S5 - VQ)

I felt that my questions weren't well thought through so the learners battled to understand what I was asking at times. This forced me to walk through every step with them with me doing a lot of the thinking for them rather than them understanding it themselves. I found myself asking "do you understand?" rather than me asking "what is your understanding?" (S6 - LR)

The preservice teachers had an opportunity to apply the knowledge they had learned in the Maths 2 course into practice in the classroom to help develop learning in and from practice (DHET, 2011b). This learning opportunity provided them with an invaluable experience of the issues and demands of teaching and the mathematical understanding needed to manage learning effectively. Through the study of practice in the form of discussion and reflection on the experience, they were able to develop their specialised knowledge for teaching further and to recognise the challenges that still needed to be overcome.

5.2.3. Specialised content knowledge in teaching an early algebra lesson – video lesson examples

The previous findings came from the preservice teachers' lesson reflections, video questionnaires and interviews and reflected their espoused knowledge for teaching early algebra. The next set of findings came from their videoed early algebra lessons which happened early in the Maths 2 course and formed the foundation for

further development of knowledge for teaching. There were five early algebra lessons across different grades selected for analysis based on the variety of the teaching tasks evident in the lesson and the diversity of issues that appeared. They were used to provide explicit examples of the complexity of tasks involved in the teaching of an early algebra lesson and opportunities for learning used or overlooked by the preservice teacher in the process of instruction.

Preservice teachers' lessons were generally organised around a set of activities that underlie functional thinking and the construction of functions for the preservice teachers: engaging in a problematic within a functional situation, creating a record and seeking patterns and mathematics certainty (Smith, 2008: 143). They worked with learner responses by encouraging learners to actively engage in the lesson. The pattern activities all represented linear functions with no mention or discussion of alternative types of patterns or different functions. There were some examples of links between the structure of the pattern and the function rule including interconnectivity between different representations of functions such as drawing, tables, flow diagrams and graphs (see Mike and Bryn in Chapter 4). Learners were also given opportunity to work with input and output values both in doing and undoing patterns. There were particular tasks which appear to challenge this group of preservice teachers in teaching early algebra across a variety of grades in the primary classroom. They did not always avail of the opportunity to link the physical objects pattern and the relationship between the numerical values recorded in the tables or flow diagrams and missed opportunities to help learners to generalise the pattern. Although the concrete and pictorial tasks helped learners to create and extend pattern, the analysis of the pattern was restricted to common difference with few links to functional thinking. These resulted in a guess and check approach to establishing the function rule and de-emphasised the relationship aspects of patterns. Some of the preservice teachers found it difficult to help learners move from recursive thinking to looking for co-variant and correspondence relationships and were hindered by their choice of representation. Using both tables and flow diagrams, in the same lesson, to build the concept of function also caused problems for both learners and preservice teachers.

5.2.4. The Maths 2 course

The Maths 2 course integrated common content knowledge and specialised content knowledge of early algebra and incorporated three purposeful events into the course

to help encourage the development of knowledge for teaching early algebra. Firstly, the preservice teachers used their teaching practicum experience to write lesson reflections which required them to think about their actions in the early algebra lesson and its contribution to the development of learners' functional thinking. This was intended to help preservice teachers experience the development of knowledge for teaching as an interactive process between theory (content) and practice. This was later followed by a structured video questionnaire to help focus their thinking around specific tasks involved in teaching early algebra and how they handled these tasks. The final event took the form of focus group interviews which happened at the end of the year, and provided another opportunity for preservice teachers to reflect and talk about their knowledge for teaching early algebra. It was hoped that through these learning and teaching experiences, knowledge for teaching early algebra would grow and flourish.

The findings related to the Maths 2 courses came from the interviews and were organised into four categories: current literature, learning community, teaching practicum and key learning principles. While there is no consensus on which learning experiences best contribute to the development of knowledge for teaching, we know learning and teaching experiences can help preservice teachers to become effective teachers of early algebra (Mc Gowen & Davis, 2001). The interview responses showed that reading and discussing literature related to early algebra and learners' relational thinking helped in the development of knowledge for teaching.

In terms of teaching the maths you can actually see how the articles have helped us to realise how to link things up and while we teach maths, what we teach in one day and not just show the kids theory (content) but to show them why it is like this and get them to explore, so that when we are maths teachers one day we would be equipped to get them to think algebraically. (FGI)

There is evidence to suggest that preservice teachers develop mathematical knowledge in "collegiate course settings" (Welder & Simonsen, 2011: 1). The feedback from the preservice teachers' interviews highlighted the need to establish and maintain a safe environment in which they could work together, feel free to ask questions and actively encourage and support one another to develop knowledge for teaching. This was a fundamental part of creating a mathematical learning

community in both professional development and classroom settings (Borko et al., 2005) and was evident in the preservice teacher feedback.

So it's almost like you can take the risk here and everybody in our class, I don't know if we just had a very nice group.....and people were actually trying to figure out and see now why did she (teacher educator) do that, okay well see like how she figured it out and whatever and then they tried to explain to you this is where you went wrong. I figured it out this is where you went wrong. (FGI)

The third category of findings from the interviews highlighted the important and crucial role of teaching practicum, otherwise known as practical learning. Mathematics field experience combined with a content/methods course appears to show the greatest promise in preparing preservice teachers to teach mathematics (Strawhecker, 2005). This combined experience helped preservice teachers to make better sense of the content from the Maths 2 course and provided a context in which to experiment and learn more about knowledge for teaching early algebra.

I think with Teaching Practice we were able to use what we've learned in the class and be able to actually put it into practice in an actual classroom and to see physical responses from the kids and how they answer questions compared to just seeing the examples and seeing the results from other studies done and now to be able to see it first for your own class. (FGI)

Lastly, there were some key learning principles which emerged from the interview responses and linked to the strands of mathematical proficiency as described by (Kilpatrick et al., 2001): conceptual understanding, strategic competence and adaptive reasoning. The preservice teachers felt that it was important for learners to have conceptual and procedural understanding of mathematical concepts and more importantly as teachers to have well established mathematical knowledge for teaching. Teachers must also be able to formulate problems and be able to devise different solution strategies.

Also in high school your motivation is just to get a good mark and to pass whereas now our motivation is to know and understand what we're learning so that we teach it because if we don't get it, sure we might fail the year but we also won't be able to teach it to the kids. It's intrinsic. (FGI)

Like K... said in high school if I had gotten that question I would never ever in a million years have thought to think like okay I have this that I know I need but I also know I need this used in an equation must give me that answer. So let me try and use the answer to try and get what I actually should have had at the beginning. I would never have thought about it like that. (FGI)

Adaptive reasoning refers to the “capacity to think logically about relationships among concepts”, to include “explanations and justifications” and are generally described as “the tools to think with” (Kilpatrick et al., 2001: 129). It was evident in the preservice feedback that they felt it was important to be able to explain and justify their thinking to others and to trial ideas, reflect on practice and grow in knowledge as they co-construct new knowledge with an expert on the content and pedagogy of early algebra (Warren, 2008).

You (teacher educator) are challenging us (preservice teachers) more to make our own conjectures and to generalise ourselves so you are not standing in front of the board and just saying this is that and we are just listening and watching. You are not telling us, you have to discuss, you have to think, you have to make your own conjectures and you have to do it yourself. It's what do you know, what do you see. Its more challenging and you learn more like that. (FGI)

The Maths 2 course and teaching practicum was designed to develop preservice teachers' knowledge for teaching early algebra. The findings from the focus group interviews helped to isolate some of the factors that contribute to this development as perceived by this group of preservice teachers. Again, it is important to remember that the illustrations from the interview responses reflect individual experiences and thoughts and cannot be generalised for the whole group or for mathematics education courses in general. What they do help to do is to isolate some of the learning and teaching experiences that can influence the development of knowledge for teaching early algebra.

5.3. Discussion

The next part of this chapter will discuss and unpack various aspects of the findings that reflect more on the issues and demands of developing knowledge for teaching early algebra for this group of preservice teachers. It will start with some concerns related to preservice teachers' common content knowledge and specialised content knowledge for teaching early algebra, followed by an overview of elements within

the Maths 2 course which are perceived by preservice teachers to be helpful in the development of knowledge for teaching early algebra. The last part of the discussion looks at the mathematical knowledge for teaching model as a design tool for the Maths 2 course and as a perspective for analysing the development of knowledge for preservice teachers' knowledge for teaching early algebra.

5.3.1. Preservice teachers' common content knowledge

Ball et al. (2008: 399) define common content knowledge (CCK) as the "mathematical knowledge and skill used in settings other than teaching". Teachers must know the mathematics they teach, be able to use correct definitions, mathematical language and notation, and recognise mathematical errors. The use of the word "common" does not imply that all people have this knowledge but it is knowledge that is not unique to teaching and can be held by others that know mathematics.

Common content knowledge, in this study relates to knowledge of early algebra. There are various entry points into EA including numerical reasoning, quantitative reasoning, and functions. The Maths 2 course was guided by the early algebra work of Blanton (2008) with inservice teachers and their learners which focused on two key areas: generalised arithmetic and functional thinking. Much of the mathematical content in the Maths 2 course focused on functional thinking with the result that most of the preservice teachers chose to teach functions through the use of patterns activities linked to physical objects, pictures, and real life contexts. The emphasis on functions and functional thinking in the Maths 2 course was to address the perceived weakness of the teaching of patterns in the primary school based on the poor results in algebra in the higher grades.

The findings from the verbal and written responses as well as the video recordings of the preservice teachers indicated a diverse range of understanding of early algebra in this particular group of teachers as indicated earlier. The preservice teachers did not appear to have difficulty with CCK: solving problems, recognising incorrect answers and using mathematical terms and notation (Ball et al., 2008). However three issues have been identified from the findings as important in using functional thinking as a route to early algebra.

Firstly, it seemed that functional thinking for many of the preservice teachers involved three activities: engaging in a problematic within a functional situation; creating a record and seeking patterns and mathematical certainty (Smith, 2008). They actively encouraged learners to study change and look for relationships to help learners to generalised mathematical ideas from a set of instances but they struggled to establish these generalisations through the discourse of argumentation (Blanton & Kaput, 2005b).

For the conclusion, I tried to get see if the learners were able to find the rule, but they were not able to, so I guided them to the right answer, which was not the absolute correct thing to do but they all did seem interested even though they were not thinking themselves. (S13 - LR)

The process of helping learners to develop functional thinking was replaced by the need to find the function rule and the role of the teacher in helping to scaffold learning through discussion was lost. Expressing generality is the most natural and fundamental mathematical activity. It starts with looking at particular cases to establish a relationship between the variables with the eventual goal of generalising. The language of algebra is then useful as it provides the symbolic system to be able to express and manipulate the generality. Mason et al. (2005) suggests that learners need to be given opportunities to find multiple kinds of patterns and that visualization and manipulation of the figures on which the generalising process is based can facilitate rule finding and formula making. Learners need to be encouraged to give detailed justifications of their generalisations with the help of the teacher. The preservice teachers make limited use of probing questions to develop functional reasoning and to scaffold in helping learners to move from the particular to the general situation. There needs to be an expectation of rigor in generalisation through learner justification as they too often gravitate towards the easiest solution without considering all options. The preservice teacher can make use of different types of functions and counter-examples to assist the development of knowledge and understanding of early algebra.

The second issue relates to the modes of analysing patterns and relationships i.e. recursive patterning; co-variational and correspondence/functional thinking which are central to the search for certainty Blanton (2008). The recursive approach emphasises the difference between the independent and dependent variable separately (looking down in the vertical function table). While the functional thinking

approach focuses on the correspondence between the independent and dependent variables (looking across the vertical function table). The process of moving between the different approaches is important for the preservice teacher in helping learners to grasp the meaning of variables and to move towards generalisation.

The learners were able to complete the function table recursively but were often not able to identify the corresponding relationship between the independent and dependent variables. The learners tended to focus on the relations among dependent variables instead of thinking about relations among sets of numbers and struggled to describe and represent relations among variables (Carragher & Schliemann, 2007). There were examples in Dani's lesson (Grade 3) and Zoe's lesson (Grade 4) in which the teacher worked hard with learners' responses to try to develop functional thinking. However there was a resistance from the learners to generalise because they did not have a need to generalise, the examples and questions asked by the preservice teacher did not require the learners to work with larger numbers. Many of the preservice teachers did not emphasise the mathematical purpose of functional thinking and over-emphasised the recursive pattern by the questions they posed to learners. Learners were required to find a function rule without establishing the need to generalise.

The focus of my lesson was identifying the common difference and recognising the pattern, and recording information in a table through the matches' activity (square) learners had to see, observe and do. Maybe if learners had to first create the 4th, 5th and 6th square with the matches on their own and see if they can identify and pick up a pattern (add 3) maybe this would have been better.
(S10 - LR)

Anthony and Hunter (2008) suggests that using tasks that involve numeric patterning and function activities can offer an opportunity to integrate early algebra thinking into the existing mathematics curriculum. However teaching algebraic concepts in the early grades requires the use of patterns to make connections using symbols to express generalisations and developing *meaning of or reason for* these symbolic generalisations (Richardson et al, 2009). It is important for preservice teachers to be aware of the need to develop meaning of and reason for generalisations. As discussed in chapter 2, there is empirical evidence to support that young children are capable of thinking functionally with the help of teachers in "using concrete materials to create patterns, specific questioning to make explicit the

relationship between pattern and its position, and specific questioning that assist children to reach generalisation with regard to the unknown positions” (Blanton & Kaput, 2004; Warren, 2005: 311). There is also evidence to suggest that learners tend to focus on pattern spotting in one data set rather than on the relationship between the pattern and its position. This may be the case either because the single variational thinking is cognitively easier for children or it could be so engrained from the school experience that there is a tendency to revert to it (Warren, 2005). This implies that some of the difficulties may not be developmental but rather experiential which makes it all the more important for preservice teachers to be aware of this and help to bridge the gaps in the early years. Limiting pattern finding to single variable data has less predictive capacity and is not as mathematically powerful as functional thinking. There is a fundamental shift in thinking needed by early grade teachers, including preservice teachers, to move from analysing single variable data (simple patterns) to two or more quantities simultaneously, in order to develop children’s functional thinking and subsequently grow their algebraic knowledge (Blanton & Kaput, 2004).

The third issue for discussion under the preservice teachers’ common content knowledge links to specialised content knowledge for teaching and deals with the teaching of functional thinking using different algebraic representations such as tables, flow diagrams, graphs and symbols. While it is important to recognise and use different representations of a function, there is a tendency to sometimes overlook the mathematical purpose of the choice of the representation in developing the concept of a function. Many of the preservice teachers’ pattern lessons included real life contexts as well as concrete and pictorial objects to represent functional relationships but sometimes opportunities were missed to expand the conceptual understanding of functions through highlighting the conceptual links between the different representations. Some preservice teachers began to see different forms of representations as easier and better representations of the function rule.

When the learners were busy with their activity, many of them struggled with writing the functional rule in words. I should have payed more attention to that. Perhaps a spider diagram (flow diagram) would have helped in this regard as the functional rule is clearly stated on the diagram. (S23 - LR)

While this preservice teacher recognised the cognitive difficulty that learners experience in trying to generalise the pattern, there was an implication that in changing the form of algebraic representation that the function rule would suddenly emerge. Common content knowledge of early algebra includes knowledge of functions and knowledge of the mathematical purpose of different forms of representations. Understanding the mathematical implications of using different representations of functions can be difficult for preservice teachers. Mason (2008) suggests that while it is possible to use objects and pictures to generate a table of values and to extract a formula from this which can be checked with one or two examples, it can lead learners to over generalise subsequent patterns too quickly and risk developing a limited concept of function. It is important in designing mathematics education course which focus on developing knowledge for teaching early algebra that over generalisation can be tacitly implied through the use of particular forms of representation of functions.

Common content knowledge is a complex concept because it initially appears to have a clear content boundary i.e. the mathematical knowledge and skills used in settings other than teaching (Ball et al., 2008). In this particular study, it is obvious that the content knowledge is related to functions but it is not obvious whether the issues raised previously are specifically related to common content knowledge of function or to some other type of common content knowledge of school mathematics. Alternatively, perhaps the issues discussed in the common content knowledge category are more suited to the specialised content knowledge category. It is not always easy to draw lines between the boundaries of common content knowledge and specialised content knowledge. Common content knowledge appears to be closely linked to subject knowledge but also subject knowledge of school mathematics and is topic specific. The common content knowledge issues that emerged from the findings are not applicable across all mathematics topics and are specific to the teaching and learning of functions. Regardless of the domain classification, the MKfT model has been useful in helping to give a perspective of this group of preservice teachers' knowledge for teaching early algebra. Through the investigation of the verbal and written responses of the preservice teachers based on the Maths 2 course and teaching practicum, a better understanding of knowledge for teaching early algebra has emerged. This study has helped to highlight the common content knowledge issues that arise in the teaching of

functional thinking for preservice teachers, and which can be used to inform the design of future mathematics education courses.

5.3.2. Preservice teachers' specialised content knowledge

Specialised content knowledge is the mathematical knowledge and skill unique to teaching and requires mathematical understanding and reasoning beyond that being taught to learners (Ball et al., 2008). Teachers use a specialised content knowledge in the everyday tasks of teaching when giving explanations, choosing examples and representations, working with learners' questions and responses, selecting and modifying tasks, and posing questions. This group of preservice teachers were developing knowledge for teaching early algebra through the study of common and specialised content knowledge for teaching in the Maths 2 course and through the act of teaching in the schools. The findings indicated that the everyday tasks of teaching early algebra drew on a specialised content knowledge which is still emerging for preservice teachers. There were different aspects of each of the tasks for teaching, as categorised by Kazima et al. (2008) that were challenging for the preservice teachers. However, there were two particular tasks of teaching which raised interesting issues and demands for further discussion in terms of transforming and connecting specialised content knowledge for teaching early algebra i.e. representations and working with learners' ideas.

a) Knowledge of representations

Representations are broadly defined to include real-life contexts, objects, drawings, as well as verbal and written symbols and expressions. Rowland et al. (2009) suggests that representations are useful to support teaching and learning as they help mediate the shift from the concrete to more abstract understanding of mathematics. Ball et al. (2004, 2008) recognise the use and understanding of representations as a unique type of knowledge required by teachers to transform content knowledge for teaching (Ball et al., 2004, 2008). This involves representing ideas carefully using different models (physical, graphical, symbolic), recognising what is involved in using a particular representation, linking and making connections between representations, and selecting presentations for particular purposes.

Mathematical ideas are enhanced through the use of multiple representations which are not just "illustrations or pedagogical tricks" but form a significant part of the mathematical content and a source of mathematical reasoning (Kilpatrick et al., 2001: 95). Some researchers suggest that through the manipulation of familiar

objects learners can build confidence in understanding structure which later emerges in the form of a generalisation, and which is likely to support algebraic thinking (Mason et al., 2005). The development of the concept of function is complex and can pose challenges for preservice teachers, especially in the selection and use of representations to support teaching and learning.

There is a great deal of research on representational systems in education and it was only natural for the preservice teachers to look for examples and representations that were familiar and confidence building when trying to help learners to make sense of something that is complex or abstract. The use of everyday contexts and concrete objects in the teaching of early algebra and the developing of algebraic thinking were both real and familiar for the learners. There were numerous examples in the video lesson recordings of a range of different representations selected to develop the concept of pattern and functions with young learners including concrete objects, pictorial diagrams, and real life contexts. The use of representations to develop mathematical understanding raised interesting questions in how they were used in this study to develop conceptual understanding of patterns and functions and how they could have been used in a different way.

There are three perspectives in the general use of representations to transform knowledge for learning, which are relevant for this research: Bruner's EIS principle; Lesh's representational system and Tall's cognitive root (Bruner, 1966; Lesh et al., 2003; Tall et al., 2000). They each approach the use of representations in different ways, some of which corresponded with the actions of the preservice teachers in the study and others which provide an alternative use of representations to develop mathematical understanding.

The most familiar and frequently used approach to representations in this study resonates with the work of Bruner (1966) and involves three sequentially developmental, but complementary forms of representations: enactive (real life objects and manipulative); iconic (drawings and diagrams) and symbolic (symbols), better known as the EIS principle. The three forms of representations can be seen in many of the preservice teacher early algebra lessons in the use of real-life examples and/or physical objects to generate numerical data, capturing the data in function tables or flow diagrams and using words and symbols to generalise relationships within the pattern. The following preservice teacher comment

illustrates a preference for working with representational modes in such a progressive sequence.

Most effective part of lesson was starting with real-life situation and also linking up the pictures to the table, the table to the formula, the formulas to the pictures and the output values on the table to the graph (S8 - LR)

Although the function table can appear to be a powerful form of representing relationships and useful for finding information about a particular value, some of the preservice teachers had difficulties in using the function table to build generalisation as it can be a poor way of understanding the overall underlying relationship in the pattern without strong scaffolding from the teachers (Mason, 2008). Some learners became fixated on the recursive pattern (or co-variate relationships) in the function table and ignored the corresponding relationship between the variables. Some of the preservice teachers tried to move between the different forms of representations to help develop relational understanding between the variable, only to cause further confusion for learners.

However, I noticed that the learners got confused when I showed them how the data in a flow diagram can be put into a table format. This was a new mathematics concept I wanted them to learn, however I think that majority of the class were not really ready for it. I should do more lessons using tables with them for future lessons to better this skill with them. (S22 - LR)

The preservice teacher comments above suggested that translating within and across representations was difficult and challenging for both teacher and learners. The Lesh translation model (Lesh et al., 2003) extends the notion of representational modes/forms and does not see representations as purely linear and progressive, but rather as interrelationships and transformations between different modes of representations (Nakahara, 2007). The model suggests that learners make more meaning when representing ideas in different but equivalent representational modes.

There are five categories of representational modes within the model: real life situations (Lesh et al., 2003); manipulatives; pictures; verbal and written responses.

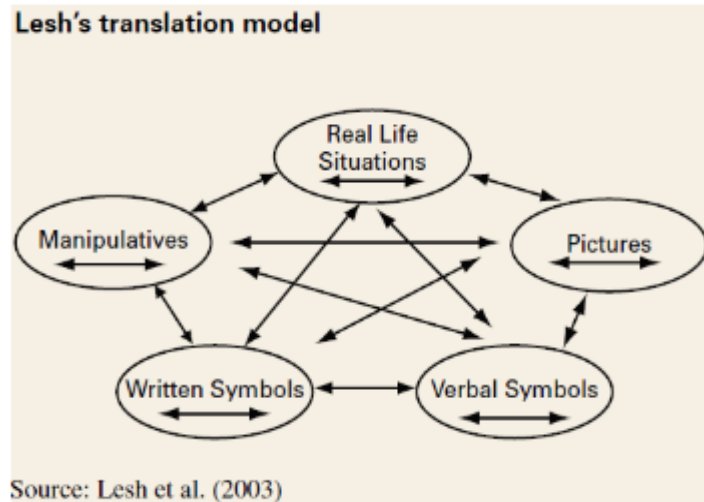


Figure 5.1: Lesh's translation model

The Lesh model is an advancement of the Bruner principle in that it emphasises a translation within and among various modes of representations which can help make mathematical concepts more meaningful for learners. Learners have the opportunity to work with patterns located in real life contexts, manipulate concrete objects such as matches or cubes and draw pictures of patterns which help develop algebraic thinking. They learn to describe and justify patterns, and to use symbols to represent pattern relationships. Chahine (2011) suggests that using multiple representations can empower learners to develop their own models when understanding other related concepts. It presents learners with opportunity to shift from strictly using procedural strategies to develop more cognitive strategies to reason and solve problems and gives them further power over their own learning. As the model capitalises on translation within and across the five representations to help develop learner understanding and internalising of mathematics concepts, it also provides opportunity for problem solving, and can ultimately improve learner performance. There were some episodes of this approach to the use of representations in the preservice teachers' early algebra lessons, as illustrated below:

Diagrams, tables and graphs were used as mathematical representations in the lesson. I used this to show how the same pattern can be represented in a different way i.e. a diagram can help us to complete a table, and the values on a table can help us to plot a graph. (S8 - VQ).

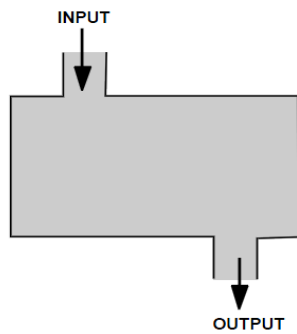
It was apparent from the verbal and written responses and video recordings that this group of preservice teachers had access to knowledge of different representational systems and applied this knowledge accordingly. However, the findings draw attention to the limiting conception of representations as tools to develop functional thinking that are held by preservice teachers. This may very well be a consequence of the design and content of the Maths 2 course but it does highlight the need for an alternative conception of function. Tall et al., (2000) extend the notion of representation to provide a powerful foundation and a cognitive root for developing understanding of the function concept. Tall et al. (2000:3) define cognitive root as a concept that:

- (i) is a meaningful cognitive unit of core knowledge for the learner at the beginning of the learning sequence,
- (ii) allows initial development through a strategy of cognitive expansion rather than significant cognitive reconstruction,
- (iii) contains the possibility of long-term meaning in later theoretical development of the mathematical concept,
- (iv) is robust enough to remain useful as more significant understanding develops.

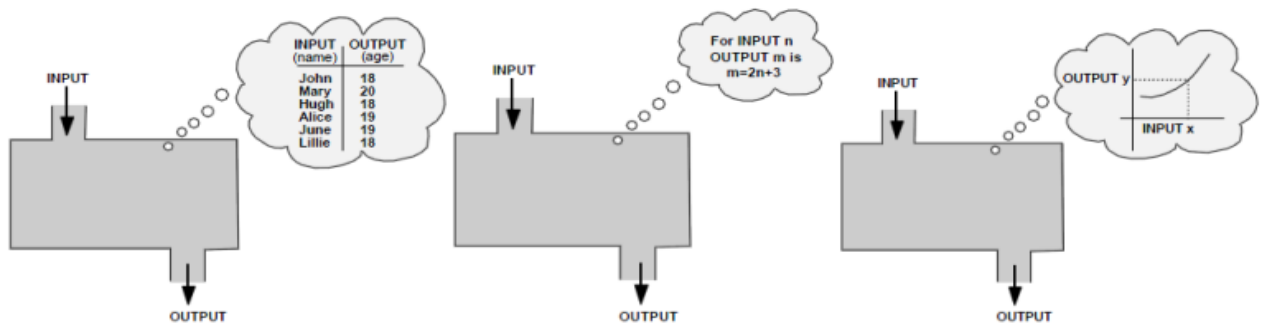
They propose the use of the function box (equivalent to flow diagram) representation to provide learners with access to the function concept as a meaningful unit of core knowledge, and helpful in building subsequent understanding about functions. It is an elaboration of the Bruner and Lesh models and provides learners with a different context from which they have to abstract the function concept. It could also help to resolve some of the issues and demands faced by the preservice teachers when using the function table and flow diagram in trying to generate a rule to describe the relationship between the input and output values.

The function box could be used as a cognitive root (anchoring concept) which exhibits as many different aspects of the function concept as possible. This involves seeing the function box as a “generic embodied image” that exhibits important aspects of the function concept which are accessible to learners yet provide a foundation for future development of the function concept (Tall et al., 2000:4). The function box is then both an embodiment of process and object duality in that it is an iconic representation of the inputs and outputs (object) while at the same time, it is a

process of representing or calculating the input and output values (process) (Tall et al, 2000:4).



The other representations of functions such as the table, formula, graph can also be seen as ways of representing or calculating the inner input-output relationship.



The function box as a table, a formula and a graph

Figure 5.2: The function box as a table, a formula and a graph

Some of the preservice teachers' responses indicated a limiting conception of the function box as used in the early algebra lessons. They saw it either as a device for calculating input and out values or as a way to help learners to find the function rule through *guess and check* of different operations. This could limit the use of the function box and prevent learners from experiencing the function box in different ways. An alternative approach would be to use the function box to promote a more general concept of a function. This could involve using the function box to represent a realistic problem, or to calculate input and output values, or to find a function rule, or to draw a graph, thereby helping learners to develop a wider understanding of the function concept more in line with the Lesh translation model.

The preservice teacher responses and actions drew attention to the use of representations to transform knowledge for teaching and learning and the issues and demands that arose in the classroom. Through the reflective comments of the preservice teachers, alternative perspectives on the role and use of representation came into focus and created the opportunity to extend the interpretation of what representations mean in the mathematics classroom. Representations as cognitive roots open up interesting possibilities in finding more cognitive roots for other mathematics concepts. Gagatsis and Elia (2005) remind us that working with representations can be challenging and success in using one mode of representation in solving a problem or representing a concept does not imply automatic success in another representation of the same concept. Different modes have different cognitive demands and distinct characteristics which makes it difficult to handle two or more representations in one mathematics task as experienced by some of the preservice teachers. Learners often see different representations of the same concept as distinct and autonomous entities and not necessarily as different embodiments of the same concept. This means that the teaching of mathematical concepts using multiple representations to create connections and comparisons cannot be left to chance and is something that needs to be taught and learned in a systematic way in teacher education courses.

b) Knowledge of working with learners' ideas

The second task of teaching selected for further discussion is working with learners' ideas which involve three different tasks of teaching (Ball et al., 2008). Firstly, it is interpreting and making mathematical and pedagogical judgements about learners' questions, solutions, problems, and insights (both predictable and unusual). Secondly, it is being able to respond productively to learners' mathematical questions and curiosities, and thirdly assessing learners' mathematical learning and taking the next step. These three tasks have been compressed by Kazima et al. (2008) to the general description of working with learners' ideas through engagement with both expected and unexpected learners' mathematical ideas.

Preservice teachers provided many verbal and written responses related to working with learners' ideas and highlighted some of the challenges it presented for them. These included managing and working with learners' responses and errors, understanding learner insights, and assessing learner understanding and planning the way forward. The preservice teachers seemed knowledgeably aware of their

shortcomings in working with learners' ideas, readily identified errors in their approach and offered suggestions for improvement in the future. Empirical research has shown that how teachers work with learners' ideas is often linked to how they hold mathematics and connected to their beliefs about why and how mathematics is learned (Fennema & Franke, 1992; Rowland et al., 2009). Barton (2009) suggests that how a teacher holds mathematics is a missing component of mathematical knowledge for teaching which includes teachers' attitudes and orientation towards mathematics. The responses of the preservice teachers reflected a conflict in their role as teachers and their beliefs about how mathematics should be taught.

Ernest (1989) and Barton (2009) identify three dimensions to teachers' beliefs: beliefs about the nature of mathematics (vision); beliefs about mathematics teaching and beliefs about mathematics learning (philosophy and orientation). Beswick (2005) provides a summary of Ernest (1989) categories of beliefs about the nature of mathematics and mathematics learning and includes Van Zoest, Jones and Thronton (1994) categories of on beliefs about mathematics teaching. This summary details three well known perspectives on the nature of mathematics: instrumentalist; Platonists and problem solving and each has its own interpretation of the impact of beliefs on teaching and learning. The instrumentalist view, otherwise known as the traditional perspective, involves the teacher as an instructor who engages in teaching mathematics for skill mastery, while learners remain as passive receivers of knowledge. The Platonists believe mathematics teaching requires teacher explanation and active construction of understanding by learners. The problem solving perspective believes mathematics is about problem solving, teacher facilitation and self-directed exploration by learners. A summary of the above follows in Figure 5.3 below (Beswick, 2005: 40):

Beliefs about the nature of mathematics (Ernest, 1989)	Beliefs about mathematics teaching (Van Zoest et al., 1994)	Beliefs about mathematics learning (Ernest, 1989)
Instrumentalist	Instructor	Skill mastery, passive reception of knowledge
Platonist	Explainer	Active construction of understanding
Problem-solving	Facilitator	Autonomous exploration of own interests

Figure 5.3: Categories of teacher beliefs

The last two categories both draw from constructivist theory and advocate construction of knowledge through different types of teacher and learner activity. Constructivism has been the “leading if not dominant theory or philosophy of learning in the mathematics education research community” for the past 30 years (Ernest, 2006: 3). Within constructivism, learning is defined as a “process of interpreting and organising information and experiences into meaningful units, transforming old conceptions and constructing new ones” (Golding, 2011: 468).

So what was the connection between teacher beliefs about mathematics, mathematics teaching and learning and working with learners’ ideas in this study? The early algebra lessons were generally organised in one of two ways: learners either worked as individuals and/or in groups depending on the activity. The expressed intention of the preservice teachers was to encourage and provide opportunity for learners to make sense of the mathematics, to critically think and engage, and to find patterns and justify their statements. They articulated an expectation and belief about mathematics teaching and learning which involved the active construction of understanding in which learners had an opportunity to understand the ‘why’ behind the mathematics and not just the ‘how’ (Ernest, 1989).

....but it’s the thinking behind the procedure that we need to stimulate and I think we were more successful in doing it with your lesson than with a numeracy lesson (focus on operations) because we are too busy focusing on the actual procedure, before you focus on getting them to critically think. (FGI)

I was trying to get them to critically think about information that seemed more abstract to them that they’re not usually exposed to. (FGI)

...we want the children to generalise their own thinking and come up with solutions for themselves and ... we want them to generalise their own theory. Rather than when you kind of giving them the theories (answers). (FGI)

The preservice teachers wanted to develop relational understanding within the early algebra lesson and move away from instrumental understanding only (Skemp, 1976). It would appear as though the pattern activity lesson provided an opportunity for the preservice teacher and learner to engage in problem solving, to experiment, to conjecture and to generalise. However there existed a dilemma in terms of how to handle learners’ responses through discussion within the early algebra lesson.

Golding (2011: 469) argues that discussions may appear to have the same form in different classrooms but there are “significant functional differences between them”. He identifies and describes three different functions of discussions: those used to help learners find or appreciate the pre-decided answers; those used to follow inquiry without a pre-decided conclusion in mind and finally discussions which give learners freedom to construct whatever ideas without constraints or structure. He believes that “functional differences are generally not taken into consideration when analysing constructivist teaching” (p. 470). The functional continuum of constructivist discussions has two opposing positions: the unstructured discussion and the teacher-directed discussion. Both extremes involve teachers in discussion with learners and learners actively involved in knowledge production. While the unstructured discussion may lead to little progress, some teachers feel the need to structure inquiry to keep learners on track towards the “teacher-determined answer” i.e. the correct answer (p.473). Some of the preservice teachers’ responses indicate an awareness of the tendency to guide the discussion and direction of the lesson too much. Unfortunately, there was sometimes so much guided instruction that the learner did not construct knowledge for themselves.

Learners were not able to identify the rule so I guided them there but I’m not sure it was the absolute correct thing to do but they all did seem interested even though they were not thinking themselves. (S13 - LR)

I should have given more time for self discovery in which the learners were given the opportunity to explore the problems on their own or in groups before addressing the problem as a class. Children learn through exploration and involvement and need to be given opportunities to make their own conjectures. I should have been less of a teacher and more of a facilitator. (S14 - LR)

One of the comments I made after the lesson was that I felt as though I’d been too prescriptive- leading the learners in their thinking rather than exploring their thoughts for fear that following their thoughts would lead to confusion. (S16 - LR)

Golding (2011) emphasises the tendency for teacher-directed discussions to lead teachers to be overly influenced by external epistemic standards such as the curriculum, established knowledge, and time constraints. This tends to privilege mathematics as a set of ‘absolute truths’ or ‘correct answers’ in which the teacher’s task is to help learners find their incorrect conceptions and reconstruct the ‘correct answer’. While it is difficult to generalise from the preservice teachers’ responses

and actions in the classroom, it is probably true to say that the development of knowledge for teaching is a development process that is both at the same time harmonious and conflicting for preservice teachers. While the intended pedagogical goal of the preservice teacher is commendable, the reality of implementation is more complex. Despite holding beliefs about the teaching and learning of mathematics which emphasised the role of learners in constructing knowledge, the preservice teachers were not always able to bring this to fruition. The early algebra lesson appeared to provide opportunity to engage learners in active learning but the preservice teachers often did not have the experience or the specialised content knowledge to be able to capitalise on this and instead resorted to teaching pattern as a set of facts and procedures.

The important learning experience from this points to the inter-related nature of knowledge for teaching early algebra. It is difficult to divorce common content knowledge and specialised content knowledge from beliefs about mathematics and beliefs about mathematics teaching and learning. What this highlights is the importance of beliefs in how we develop knowledge for teaching early algebra, how we learn and teach early algebra, and the crucial role beliefs need to play in the design and implementation of teacher education courses.

5.3.3. Maths 2 course

There are different approaches to mathematics and mathematics education courses for teachers. Some programs teach the mathematics first and apply that knowledge to teaching which could mean that some teachers can do the mathematics but are unable to apply it in teaching. Thames et al. (2008) argue for an alternative approach which involves integrating and learning the mathematics required for teaching in the contexts of its use in practice. The Maths 2 course uses an integrative approach and combines knowledge of functions and knowledge for teaching functions to develop preservice teachers' knowledge for teaching early algebra. This study provides a vehicle for understanding how the Maths 2 course is useful in helping develop knowledge for teaching early algebra and also highlights issues that need further consideration.

There are three elements of the course which appear to be helpful in developing knowledge for teaching early algebra as indicated by the preservice teachers in the focus group interviews i.e. exposure to empirical research literature on early

algebra; working in a learning community; and teaching practicum. The preservice teachers' written and verbal responses indicate that the course has been useful in exposing them to empirical research in early algebra involving young children and functional thinking. It has helped the preservice teachers to grapple with issues related to transforming knowledge of early algebra into the classroom and built an awareness of research in the field of early algebra. They read and learned more about young children's capacity for algebraic thinking and it appeared to give them confidence to apply their knowledge in the classroom.

The second element of the Maths 2 course design focused on creating a learning community in which preservice teachers could learn mathematics for teaching in a safe environment, solving and discussing rich tasks in a shared process of learning. Many preservice teachers enter teacher education with a strong procedural understanding of mathematics which is insufficient for teaching. The Maths 2 course was designed to provide opportunity for preservice teachers to unpack their knowledge of functions and to challenge one another's assumptions. It was important to create a safe environment in which to explore their mathematical understanding of algebra and to develop their knowledge for teaching early algebra. They appeared to recognise and appreciate the support of one another in the Maths 2 course and begun to experience the opportunity of working in a safe environment.

The final element was the application of the knowledge of mathematics and knowledge for teaching mathematics in the classroom in the form of the teaching practicum. This provided the chance for the preservice teachers to engage with early algebra in the context of its use in practice and helped to expose some of the issues and demands of applying knowledge in teaching. There were different levels of cognitive engagement in the early algebra lessons which exposed the diversity of knowledge for teaching early algebra within the preservice teachers group. There were conceptual knowledge issues related to the development of functional thinking for some and there were challenges in giving explanations, using representations and working with learner responses for others. The lesson reflections, questionnaires and interview responses were greatly enhanced through the teaching practicum experience and provided a context for preservice teachers' to reflect on their practice. The structured opportunities for reflection in the form of class discussions, post lesson interviews, questionnaires and interviews happened

throughout the course and appeared to help the preservice teachers in the development of knowledge for teaching early algebra.

The preservice teachers' responses highlight the important role of building both conceptual and procedural understanding of functional thinking and the challenges that arise when transforming this knowledge for learners. The issues that emerge are useful in informing teacher education courses and give specificity to the demands of developing mathematical knowledge for teaching. It reinforces the need to support preservice teachers in shifting from understanding knowledge of school algebra towards developing knowledge for teaching early algebra.

5.3.4. Mathematical knowledge for teaching

Various models of teacher knowledge are available which describe components of teacher knowledge and are useful for investigating different aspects of teaching. Currently, there is no agreement on a model of teacher knowledge that best describes teachers' mathematical knowledge for teaching (Goulding & Petrou, 2008). There is no doubt that mathematical knowledge for teaching involves a "complex combination of different types of knowledge that interact with each other to inform teaching" (Rowland et al., 2009: 26). This study chose to use the Mathematical Knowledge for Teaching (MKfT) model to design an early algebra course (Maths 2) and to analyse the verbal and written responses and video-recorded lessons of preservice teachers, to understand their development of knowledge for teaching early algebra. The comments which follow briefly revisit the definition and features of the MKfT model and its applicability to this study. This is followed by a discussion of some aspects of mathematical knowledge for teaching which arose from this study. This section (5.3.4) concludes with a discussion of recent empirical research which provides a promising perspective on the relationship between common content knowledge (CCK) and specialised content knowledge (SCK) for teaching.

The MKfT program of research was founded in part as a result of Ball's interest in codifying a theory of content knowledge and Hill's desire for a set of instruments that could be used to evaluate professional knowledge (Hill et al., 2012). MKfT is defined as the knowledge, skills and sensibilities (habits of mind) needed to support teachers as they engage in the task of teaching. The MKfT model seeks to elaborate the domains of knowledge from the practice of teaching and helps in

attending to the mathematics and the work of teaching to provide a perspective on teacher knowledge and its relationship to teaching (Even, 2009).

The MKft model was useful in this study to help focus on the development of preservice teachers' common content knowledge (CCK) and specialised content knowledge (SCK) for teaching early algebra. It highlighted the issues and demands of teaching for preservice teachers and provided a helpful heuristic for considering types of knowledge needed for teaching. It gave insight into the complexity of developing knowledge for teaching early algebra by focusing on the tasks of teaching and the demands these tasks presented for preservice teachers.

There were two interesting issues which arose in relation to using the model to understand the development of knowledge for teaching. The first relates to the classification of knowledge into CCK and SCK. It was sometimes difficult, in this study, to classify the verbal and written responses exclusively into CCK and SCK, as often the comments seemed to include aspects of both types of knowledge. This was also apparent in the examples given in the case studies of the early algebra lessons. It suggests that knowledge for teaching is not easily restricted to specific domains and that the boundaries between the domains can often be blurred. It confirms suggestions by Ball, Hill & Bass (2005: 45) that it is still an open question "whether specialised content knowledge for teaching mathematics exists independently from common content knowledge".

The second issue relates to the composition of common and specialised content knowledge for teaching. CCK is defined as the mathematical knowledge and skills used in settings other than teaching, and SCK as the mathematical knowledge and skills unique to teaching. These categories emerge from the analysis of the practice of teaching; however, preservice teacher education is more concerned with the development of knowledge for teaching. Although the model provides broad definitions of what constitutes CCK and SCK, it does not seek to identify the topic-specific common and specialised content knowledge needed for teaching. What this study begins to do is to elaborate the topic-specific common and specialised content knowledge that is needed by preservice teachers for teaching early algebra. More research is needed on what constitutes CCK and SCK for teaching specific topics in preservice teacher education.

Bair & Rich (2011) raise an interesting point about the assumption that is often made that CCK is a necessary precondition for advancement in SCK. Their research with teacher education (TE) students showed that those with low levels of CCK to solve problems could show some growth in SCK, within the same content. They also found that TE students can work at “simultaneously developing lower-level components of SCK as they developed their CCK but could not move to “higher levels of SCK without CCK” (p.315). Traditionally in South Africa, preservice teachers enter teacher education with low levels of CCK which could make it difficult to develop higher levels of SCK for teaching. The results of the Bair & Rich (2011) research offer hope for the simultaneous development of CCK and SCK and the possibility of development of MKfT in the limited time available for teacher education. This could mean that teacher education courses which offer both CCK and SCK could help develop preservice teachers’ knowledge for teaching.

5.4. Conclusion and recommendations from this study

The following section draws together concluding thoughts relating to teacher knowledge, early algebra as functional thinking and professional development of teachers. There is also some discussion of the implications of this study and possible recommendations for future research.

5.4.1. Teacher knowledge

Teacher knowledge is a complex and dynamic construct which contains many dimensions and domains. Ball et al. (2008) have engaged in the challenging task of classifying knowledge and describing domains of mathematical knowledge needed for teaching through examining the practice of teaching in the classroom. Not all agree with the Ball model’s boundary demarcations and point to the obvious overlaps in certain categories of knowledge. There is fuzziness between the boundaries which makes it difficult to categorise knowledge exclusively to one domain (Petrou & Goulding, 2011). It is not always easy to decide when knowledge is purely subject matter knowledge (SMK) or pedagogical content knowledge (PCK) or both. Notwithstanding the contestations, the work of the Ball team and others present different and interesting perspectives on mathematical knowledge for teaching which provide various options for the interrogation and investigation of practice (Davis & Simmt, 2006; Adler & Pillay, 2007; Ball et al., 2008; Rowland et al., 2009; Watson & Barton, 2011).

This study sought to understand the development of preservice teachers' knowledge for teaching early algebra using Ball's mathematical knowledge for teaching (MKfT) model, with a particular focus on common content knowledge (CCK) and specialised content knowledge (SCK) for teaching early algebra. It made use of the model to analyse the written and verbal responses and actions of preservice teachers which was a new and different application of the model to understand the development of knowledge for teaching early algebra. The model helped expose issues which arose in developing CCK such as functional thinking and generalisation, modes of analysing pattern such as recursion and correspondence, and the mathematical purpose of functional representations. This knowledge is specific to the topic of functions and particularly relevant to the work of teaching early algebra. The findings also highlighted aspects of SCK for teaching early algebra which were problematic for this group of preservice teachers, more especially tasks such as giving explanations, using representations, questioning and working with learners' ideas.

Knowledge is traditionally compartmentalised in teacher education courses, yet practice shows that CCK and SCK are often indistinguishable (Barker, 2007). Preservice teachers come with under-developed knowledge for teaching and lack experience in integrating various knowledge forms. The more exposure they can get to courses which offer a blend of knowledge, the more likely they are to feel more competent to teach mathematics. This study shows that integrating CCK and SCK in mathematics education courses offers rich possibilities for the development of knowledge for teaching. It helps preservice teachers to begin to engage with the issues and demands of learning and teaching early algebra and to become aware of the different types of knowledge needed for teaching.

There are many different aspects of teacher knowledge that arose from this study which could be investigated further. Two aspects were particularly pertinent: knowledge of content and students and beliefs of preservice teachers. Knowledge of content and students is part of pedagogical content knowledge (MKfT model) and was not explored in this study but is an integral part of the teaching and learning process. The focus of this study was on the development of preservice teachers' common and specialised content knowledge for teaching early algebra. However, knowledge of content and students and the strategies students use to generalise pattern activities are important. It is also vital to understand what influences

students to use certain strategies and the role of the teacher in helping students to develop different ways of solving algebraic problems. Duncan and Schafer (2007) highlight the issue of question design in influencing the strategies that learners use to solve pattern generalisation tasks. They found that while pictorial context might be useful for some learners, it can cause additional complications for others. The second aspect that emerged strongly in the study was the role of preservice teachers' beliefs about mathematics and mathematics teaching and learning, and its impact on their behaviour in the classroom. There is already some research on the link between beliefs and teacher knowledge which resonate with this study (Ernest, 1989; Fennema & Franke, 1992; Beswick, 2005). Future research possibilities could investigate beliefs of preservice teachers and its link to SCK through the investigation of their actions in the classroom. Understanding the link between behaviour, belief and attitudes could be an important component in the development of knowledge for teaching.

5.4.2. Early algebra as functional thinking

Early algebra is a relatively new domain within mathematics education but it is a growing and interesting topic that offers a different approach to teaching and learning algebra. It integrates arithmetic thinking and algebraic thinking in the lower grades to help learners develop a conceptual understanding of algebra. There are many different aspects of early algebra and different entry points. This study elected to use functional thinking as an entry point and used this to develop preservice teachers' knowledge for teaching early algebra. This study has helped to unpack some of the critical issues in using functional thinking as a route to early algebra through the investigation of pattern activities with preservice teachers. This knowledge is not exclusive to preservice teachers, and it is important for all teachers to understand the purpose of such activities and the possibilities they present. Blanton (2008) highlights the power of functional relationships in the primary grades to allow learners to calculate the value of any point without knowing the previous value. This develops further in the later grades when learners learn to analyse functions and graphs in terms of how they grow and change thus building the foundation for formal algebra and calculus. All teachers, including preservice teachers, need to understand the concepts and processes that guide this development. The findings of the preservice teachers' responses and lessons alert us to the crucial role of the teacher in developing functional thinking through effective questioning and scaffolding. Teachers also need to understand the role of

representational systems in providing explanations and for creating opportunities to expand the concept of function.

This has implications for curriculum implementation in terms of how we approach the transition from working with patterns in the Foundation Phase (FP) to the Intermediate and Senior (ISP) Phase level. The FP curriculum guidelines specify the use of patterns in the younger grades, starting with concrete, pictorial and real-life contexts to create and extend patterns. This changes in the Intermediate Phase to using pattern activities to emphasise relationships within the data. It is a problem when pattern activities remain as extensions of arithmetic activity and are not used to build an intuitive understanding of relationships between numbers and ultimately build functional thinking. This is evident in the many pattern tasks available in worksheets and textbooks at the Foundation Phase level which require learners to extend the pattern only (recursive patterning) without looking for other relationships. Mason (2008) suggests that curricula and teachers need to exploit and make use of children's natural power to express generality. Teachers need to develop learners' ways of thinking through processes such as "guess and test"; "try and improve", "spot and check"; and "using structure" (p. 85). He believes there is little point in providing particular cases using pattern activities, displaying methods and worked examples and expecting learners to suddenly be able to generalise. Teachers need to explicitly and intentionally prompt learners to use their powers to generalise and have opportunity to regularly use this power in the mathematics classroom from an early age.

Teacher professional development at the preservice and inservice levels needs to address these issues in helping teachers to interpret and implement the curriculum requirements for patterns and functions. This necessitates an investigation of ways to help teachers provide "rich and connected experiences in algebraic thinking" and to help teachers develop "algebra ears and eyes" (Blanton & Kaput, 2003: 70). This could involve helping teachers to recognise algebraic thinking in their daily mathematics lessons, designing and trialling algebraic activities with learners, finding links to the curriculum and sharing ideas with other teachers working in different contexts. If we want to promote and develop algebraic thinking in the primary school, it will require a concerted effort to mobilise teacher educators, teachers and curriculum advisors to rethink our current approach to algebra teaching.

There is an opportunity to promote early algebra in schools through the effective use of preservice teacher education programs. This study shows that early algebra creates opportunity for preservice teacher and learners to engage in algebraic thinking as part of pattern activities. The Maths 2 course and teaching practicum gave preservice teachers the chance to put their specialised content knowledge for teaching early algebra into practice. Through the planning and teaching of additional early algebra lessons, preservice teachers can begin to develop their algebra ears and eyes. Through having their own experience of a rich and connected experience of algebra, preservice teachers can have a better understanding of how to build these opportunities with their learners (Blanton & Kaput, 2003).

5.4.3. Professional development of preservice teachers

Preservice teacher education is about creating opportunities to develop knowledge and understanding of mathematics itself and knowledge of mathematics pedagogy. It recognises that preservice teachers come with a mix of different school experiences of learning mathematics, and often have content knowledge that is under-developed for teaching. They need help in developing knowledge for teaching through the study of common content knowledge and specialised content knowledge. The findings of this study indicate that there were three aspects of the Maths 2 course and teaching practicum which were useful in the development of knowledge for teaching: creating a learning community, linking empirical research on early algebra and learners, and integrating teaching experience. The learning community provided the foundation for collaborative learning of mathematics and mathematics pedagogy in a safe and nurturing environment which encouraged and challenged preservice teachers to develop their knowledge for teaching. Creating opportunity for guided reflection and discussion throughout the Maths 2 course, linked to teaching practicum, helped to develop preservice teachers who can interrogate their knowledge for teaching in useful ways. They learned to recognise their strengths and weakness and developed the disposition to search for answers and for ways to improve their practice. Having the opportunity to integrate theory (CCK and SCK) and practice (teaching practicum) and to reflect on the process gave the preservice teachers time and space to develop knowledge for teaching. They learned from their own learning and teaching experiences to deal with challenges and to look for alternatives.

There is a strong case to be made for the integration of disciplinary, pedagogical, practical and situational learning in mathematics education courses. The findings of this study appear to indicate that giving preservice teachers their own experiences of developing knowledge for teaching early algebra helps build understanding of how to create these opportunities for learners. Barker (2007) suggests that learning knowledge and using knowledge are not separate processes. If learning and using knowledge are both part of developing knowledge for teaching then preservice teacher education courses need to create closer links between mathematics, mathematics education courses and teaching practicum. There is a need to rethink mathematics education course curricula to include specialised content knowledge for teaching in addition to sound common content knowledge.

5.5. Final thoughts

Preservice teacher education is primarily concerned with developing proficiency of teacher knowledge in terms of content knowledge, pedagogical knowledge and pedagogical content knowledge to ultimately improve the proficiency of learners in schools (Lerman et al., 2009). According to Ball (1988) preservice teachers do not come to teacher education knowing nothing about mathematics, about the teaching and learning of mathematics and about schools. They have already formed an interconnected web of ideas and deep-rooted beliefs based on their personal experiences in the classroom. The challenge of teacher education is to reshape these beliefs and correct misconceptions to promote effective teaching in mathematics. Teacher education is in a unique position because in the process of learning to become a mathematics teacher, preservice teachers have to be both student and teacher. During the process of acquiring the mathematical knowledge for teaching to become effective teachers, they also have the opportunity to “recast their initial beliefs about what it means to teach, what it means to learn and even what it means for something to be mathematics” (Liljedahl et al., 2009: 29). The Maths 2 course and teaching practicum have given these preservice teachers the chance to develop both the common content knowledge and specialised content knowledge of early algebra while at the same time giving them opportunity to explore its translation into the teaching context. It has also provided a context in which to begin to engage with their deep-rooted concepts of what it means to learn algebra. The preservice teachers have started the journey of beginning to integrate different forms of knowledge with their experience of learning and teaching early algebra to form their own personal knowledge for teaching early algebra.

This single course cannot hope to create permanent change in the preservice teachers' perceptions of mathematics and mathematics teaching, nor effect lasting changes in classroom practices because, as we know, schools are highly effective in absorbing new teachers into the prevailing culture of the school (Gellert et al., 2009). However, what the Maths 2 course tries to do, is to develop a more "reflective mathematics education culture" in which teacher educators and preservice teachers begin to engage with the conceptualisation of knowledge needed for teaching early algebra and hopefully other topics thereafter (p. 54). Through guided support and reflection, teaching practicum can become a contributing factor to developing knowledge for teaching instead of a continuation of existing practices of compartmentalising knowledge. The Maths 2 course has given the preservice teachers opportunity to gain knowledge of practice through participation in practice and reflection on practice and to help create a reflective disposition (Matos et al., 2009). It remains to be seen whether this group of preservice teacher will be able to maintain the practice and to continue to develop their knowledge for teaching early algebra.

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APPENDIX A



SUBJECT GUIDE 2010

FACULTY:	Education, Mowbray campus
DEPARTMENT:	General Education & Training (GET)
COURSE:	BEd 3: GET Foundation Phase BEd 3: GET Intermediate and Senior
Phase SUBJECT:	Mathematics 2
ITS CODE:	MMS 210R (FP & ISP major) LRA31RA (FP non-major – Semester 1 course)
LECTURER:	
OFFICE:	O,71
PHONE:	021 - 680 1528 (w)
MODERATOR:	

PURPOSE OF THE COURSE

To equip students with the necessary knowledge, understanding and skills to teach mathematics competently and confidently in the GET Phase.

SUBJECT COURSE OUTCOMES

- To develop students understanding of algebraic thinking and reasoning as:
 1. Generalised arithmetic
 2. Meaningful use of symbols
 3. Study of structure in the number system
 4. Study of patterns and functions
 5. Process of modelling, integrating the first four list items
- To expand knowledge of measurement through investigative activities
- To derive and use rules for calculating measurements relating to geometric figures and solids
- To apply data handling knowledge and skills to investigate and solve problems
- To recognize the difference between the probability of outcomes and their relative frequency in simple experiments
- To investigate chance (probability) as single and compound events

PROGRAMME

Topic/Module/Unit	Date	Lecturer(s)
1. Number Patterns and Algebra	27 Jan – 19 Feb	Sharon Mc Auliffe
2. Algebra as generalised arithmetic	22 Feb – 06 Mar	
3. Functions and graphs	08 Mar – 28 Mar	
4. Global graphs	10 May – 21 May	
5. Measurement in 2D and 3D	24 May – 03 Sept	
6. Data Handling (MMS210R only)	15 Sept – 01 Oct	
7. Probability (MMS210R only)	04 Oct – 29 Oct	

ASSESSMENT

Assessment is continuous which means that your progress is evaluated through the year by means of modular tests, class tasks/assignment **as well as** a final test which is written during October/November.

MMS210R Students:

Final mark	Weighting	Mark entered on MAS date
The year mark consists of:		
Modular test 1 (19 March)	20%	30 April 2010
Class tasks/Assignment 1 & 2/Journal	20%	30 June 2010
Modular test 2 (1 Oct)/Assign 3/Journal	20%	31 October 2010
Final Test (ALL modules)	40%	30 November 2010
Total Mark	100%	

LRA31RA Students:

Final mark	Weighting	Mark entered on MAS date
The year mark consists of:		
Modular test 1 (19 March)	20%	30 April 2010
Class tasks/Assignment 1 & 2/Journal	20%	30 June 2010
Modular test 2/Journal (3 Sept)	20%	31 October 2010
Final Test (ALL modules)	40%	30 November 2010
Total Mark	100%	

To pass the subject, you need to obtain a **final mark** of at least 50%. If your final mark is between 45% and 49% you qualify for a re-assessment. Names of such students will be posted on the notice board after the exams. It is **your responsibility** to consult the notice board to determine whether you qualify for a re-assessment.

SUBJECT CONTENT

Algebra (LO2: Patterns, functions and algebra):

The student should be able to

- generalise numerical and geometrical patterns (i.e. identify the pattern, determine the rule and find the n-th term);
- describe numerical and geometrical patterns, including the Figurate numbers, in words and symbolically;
- construct and solve linear equations taken from different contexts;
- apply induction and deduction reasoning to develop the concept of algebra as generalized arithmetic;
- be able to describe the relationship between two variables symbolically, and operate on these to determine values of the variables (substitution and solution of equations);
- understand functions as relationships between variables, and to use function terminology confidently;
- draw accurate graphs of linear and non-linear functions from descriptive, tabular and symbolic information with meaning;
- draw and interpret sketch graphs of linear and non-linear functions (in symbolic and descriptive modes);
- understand, apply and interpret the concept of gradient and average rate of change;
- draw and interpret global graphs;
- integrate the different representations of algebraic reasoning

Measurement (LO4):

The student should be able to.....

- change the subject of formulas and complete relevant calculations using the formulas
- prove and apply area formulas of quadrilaterals
- determine formulas and calculate the values of the surface area and volume of two- and three-dimensional solid objects;
- solve measurement problems in a variety of contexts

Data-handling, including probability (LO5):

The student should be able to.....

- conduct and present a statistical survey
- draw, interpret and select the most appropriate type of statistical graph to represent information
- calculate and use measures of central tendencies and variance effectively in different contexts
- be able to draw and interpret frequency curves
- understand basic principles of probability and be able to apply them
- recognise the difference between the probability of outcomes and their relative frequency in simple experiments

- investigate chance (probability) as single and compound events
- use tree diagrams to illustrate the multiplicative principle

METHODOLOGY AND TIME MANAGEMENT

1. Teaching method

Instruction will include a combination of lectures, practicals, individual and group work, with ample opportunity for self-activity and discussion.

2. Attendance and Punctuality

One hundred percent attendance is required from you. The nature of the subject, and the way in which we present it, is such that attendance of **ALL** students is required. Some year marks may be allocated for attendance and participation. In case of illness or any valid reason, absence **MUST** be explained to the lecturer as soon as you are back in class.

3. Calculators

You will need a calculator for your class work and you have to bring it to every class. (Casio: fx-82ES PLUS)

4. Homework

Students do not make progress unless they do tasks set for homework. We expect each student to spend at least 1 hour per week on homework. It is important that students are well prepared when they come to class.

LEARNER MATERIALS

No textbook is required. Material will be made available in the form of summarized notes and transparencies. You will be required to make your own notes in class. You must obtain a file for use in this subject only. Please insert this subject guide and all class notes into the file. All personal notes taken from the board as well as examples, should also be placed in the file. Homework, spot tests, modular tests and other class tasks should also be placed in the file.

You will be required to keep a journal of your mathematical development over the year and a mark will be allocated to the depth and explanation of your reflections.

STUDENT SUPPORT

1. Tutor program

The university provides a peer tutor system to support mathematics students, free of charge.

If you should require extra help in mathematics, contact the tutor co-ordinator, telephone numbers available on the mathematics notice-board outside lecture room 1.16.

2. Computer Package

The complete Master Maths programme is installed on the faculty computer network and can be accessed from the students computer workroom. The Master Maths program offers a complete tuition program of the school mathematics curriculum Grade 1-12 and is based on the current school curricula. To access the program, you first need to register with the technical assistant in the computer room.

3. Library Resources

There is a file of past exam papers available in the library for students to copy as well as numerous maths textbooks which should be consulted if a student experiences any problems.

TIME ALLOCATION

The course runs for the whole year, i.e. 23 weeks. Three periods per week are allocated for mathematics content. **LRA31RA students finish on the 3 September 2010 (Week 18).**

SUBJECT GUIDE 2010

FACULTY:	Education, Mowbray campus
DEPARTMENT:	General Education & Training (GET)
COURSE:	BEd3: GET Intermediate and Senior Phase
SUBJECT:	SPECIFIC SUBJECT DIDACTICS: LEARNING AREA 3 - MATHEMATICS
ITS CODE:	SLA31RA
LECTURER:	
OFFICE:	0,71
PHONE:	021 - 680 1528 (w)
MODERATOR:	

PURPOSE OF THE COURSE

To prepare students to teach and assess mathematics in the primary school, using a problem-solving approach through the use of a variety of different media.

SUBJECT COURSE OUTCOMES

At the end of this course, students should be able to:

- To develop the knowledge of how to teach mathematics through problem-solving approach in the primary school.
- To identify the role of group work and co-operative learning in the mathematics classroom as well as the role of practice in problem-based classrooms.
- Develop an understanding of how to develop algebraic thinking, measurement, data handling and probability concepts in ISP and to plan suitable learning tasks.
- Use and apply the National Curriculum Statement (NCS) for Mathematics, with emphasis on the Intermediate and Senior Phase.
- Demonstrate understanding of how to build assessment into mathematics teaching and learning.
- Design and implement mathematics assessment tasks for the primary school.

PROGRAMME

Topic/Module/Unit	Date	Lecturer(s)
1. Teaching through problem-solving and planning in the problem-based classroom 2. Learning and teaching of algebraic thinking & technology 3. Learning and teaching of measurement 4. Assessment 5. Learning and teaching data handling and probability		

ASSESSMENT

Assessment in this subject takes place on a continuous basis.

A minimum final mark of 50% must be obtained in order to pass the subject.

No re-evaluation can be obtained in this subject.

Class attendance may influence a student's final mark.

Assessment activities:

Activity	Date	Topic	Relative weight
Assignment 1	04 June	Algebraic thinking	50%
Assignment 2	15 October	Assessment	50%

SUBJECT CONTENT

1. Teaching through problem solving and Planning in the problem-based classroom

The student should be able to:

- Demonstrate understanding of the shift from the rule-based, teaching by telling approach to a problem-solving approach to mathematics teaching and illustrated with mathematics examples.
- Outlining a step-by-step approach for a problem-based lesson, and identify the role of group work and co-operative learning in the mathematics class, as well as the role of practice in problem-based mathematics classes.
- Identify different forms of assessment
- Design a variety of assessment task for algebra, geometry and probability
- Present different forms of assessment feedback to learners

2. Learning and teaching of Algebraic thinking (LO2)

The student should be able to:

- Develop an understanding of how Algebraic thinking develops in the Senior phase.
- Identify and interpret the knowledge and skills needed for Algebra as prescribed by the National Curriculum Statement for SP.

- Plan and demonstrate various strategies to teach Algebra based on research and models for learning.
- Critically evaluate Mathematics materials for suitability in teaching Algebra.

3. Learning and teaching of Measurement (LO3/4)

The student should be able to:

- Demonstrate an understanding of children's development of measurement ideas for Senior phase.
- Identify and interpret the knowledge and skills needed for Measurement as prescribed by the National Curriculum Statement for SP.
- Plan and design a learning trajectory for Measurement in the Senior Phase.
- Critically evaluate Mathematics materials and models used in the teaching of Measurement.

4. Assessment

The student should be able to:

- Explores outcomes-based assessment of mathematics in terms of five main questions
 - Why assess? (the purposes of assessment);
 - What to assess? (achievement of outcomes, but also understanding, reasoning and problem-solving ability);
 - How to assess? (methods, tools and techniques);
 - How to interpret the results of assessment? (the importance of criteria and rubrics for outcomes-based assessment) ; and
 - How to report on assessment? (developing meaningful report cards).

5. Learning and teaching of Probability (LO5)

The student should be able to:

- Develop an understanding of how Probability develops in the Senior phase.
- Identify and interpret the knowledge and skills needed for Probability as prescribed by the National Curriculum Statement for SP.
- Plan and demonstrate various strategies to teach Probability based on research and models for learning.
- Critically evaluate Mathematics materials for suitability in teaching Probability.

METHODOLOGY AND TIME MANAGEMENT

1. Teaching method

Instruction will include a combination of lectures, practicals, individual and group work, with ample opportunity for self-activity and discussion.

2. Attendance and Punctuality

One hundred percent attendance is required from you. The nature of the subject, and the way in which we present it, is such that attendance of **ALL** students is required. Some year marks may be allocated for attendance. In case of illness or any valid reason, absence **MUST** be explained to the lecturer as soon as you are back in class. We have a set programme and we plan to finish the programme. Lectures will start punctually.

3. Homework

Students are expected to complete homework assignments as discussed during class time.

4. Learner materials

No textbook is required. Material will be made available in the form of summarized notes and transparencies. You will be required to make your own notes in class.

You must obtain a file for use in this subject only. Please insert this subject guide and all class notes into the file. All personal notes taken from the board as well as examples, should also be placed in the file. Homework, spot tests, modular tests and other class tasks should also be placed in the file.

5. Time allocation

The course runs for the whole year, i.e. 23 weeks. One period per week is allocated for mathematics pedagogy (didactics).

APPENDIX B

BED3: MATHS 2 (FP and ISP): Teaching Practice

Assignment

Assignment Purpose

- To design and teach a mathematics lesson based on algebraic thinking (early algebra)
- To write a reflection based on the post-lesson interview
- To analyse the video recording of your early algebra lesson using a set of guiding questions (video questionnaire)
- To write a set of answers to the questionnaire substantiating comments from lecture notes, class readings and discussion

Assignment Instructions

1. Prepare and plan a lesson (30 – 45 minutes) on algebraic thinking (LO1 & 2), refer to activities discussed in class.
2. Write a detailed plan of the lesson (lesson plan) including time allocation, questions for learners and lesson activities.
3. Discuss the lesson content and teaching approach with your tutor teacher for input and guidance.
4. Make appointment with the mathematics lecturer to have the lesson videoed
5. You will have a short interview after the lesson to discuss the pros and cons of the lesson.
6. You will be given a DVD copy of the lesson after teaching practice and you will be required to analyse the lesson based on a set of criteria supplied by the lecturer. You will need to prepare a reflective essay based on your understanding of the theory studied in class and on your practical experience in the classroom.

NOTE: Please remember that all information is confidential.

Assessment criteria

- Lesson preparation and planning including all notes and activities
- Lesson analysis and reflection
- Linkage to lecture notes and class discussion

APPENDIX C

Navrae Dr RS Cornelissen
Enquiries
IMibuzo

Telefoon (021) 467-2286
Telephone
IFoni

Faks
Fax (021) 425-7445
IFeksi

Verwysing
Reference 20091204-0035
ISalathiso



Wes-Kaap Onderwysdepartement

Western Cape Education Department

ISEBE IeMfundo IeNtshona Koloni

Ms Sharon Mc Auliffe
Faculty of Education and Social Sciences
Cape Peninsula University of Technology
MOWBRAY
7700

Dear Ms S. Mc Auliffe

RESEARCH PROPOSAL: DEVELOPING PRESERVICE TEACHERS' CONTENT KNOWLEDGE FOR TEACHING EARLY ALGEBRA.

Your application to conduct the above-mentioned research in schools in the Western Cape has been approved subject to the following conditions:

1. Principals, educators and learners are under no obligation to assist you in your investigation.
2. Principals, educators, learners and schools should not be identifiable in any way from the results of the investigation.
3. You make all the arrangements concerning your investigation.
4. The programmes of Educators are not to be interrupted.
5. The Study is to be conducted from **18th January 2010 to 30th September 2010.**
6. No research can be conducted during the fourth term as schools are preparing and finalizing syllabi for examinations (October to December).
7. Should you wish to extend the period of your survey, please contact Dr R. Cornelissen at the contact numbers above quoting the reference number.
8. A photocopy of this letter is submitted to the principal where the intended research is to be conducted.
9. Your research will be limited to the list of schools as submitted to the Western Cape Education Department.
10. A brief summary of the content, findings and recommendations is provided to the Director: Research Services.
11. The Department receives a copy of the completed report/dissertation/thesis addressed to:

**The Director: Research Services
Western Cape Education Department
Private Bag X9114
CAPE TOWN
8000**

We wish you success in your research.

Kind regards.

Signed: Ronald S. Cornelissen
for: **HEAD: EDUCATION**
DATE: 15th December 2009

18 March 2010

Dear Principal and Tutor teacher

Permission to video-record a Mathematics lesson based on algebraic thinking

The Department of Mathematics Education at CPUT's Faculty of Education and Social Sciences has started a research project into **Algebraic Thinking**. In the first stage of the project, we are working with our BEd3 Mathematics specialization students in the Foundation and Intermediate Phase. We have spent the past term looking at the theory, research and implementation of algebraic thinking lessons (early algebra as it is otherwise known) in the primary school.

The importance of Algebraic Thinking, as a field of study, is internationally recognized and implemented. In South Africa it forms part of Learning Outcome 2 (Patterns, Functions and Algebra) of the Revised National Statement for Mathematics. Since Algebraic Thinking is a relatively recent concept in South Africa, only introduced with the NCS, we as teacher educators wish to prepare our student teachers effectively to teach this important topic. We would like the BEd3 students to have to opportunity to design and implement a mathematics lesson based on algebraic thinking. The tutor teacher will be given the detailed lesson before implementation for input and feedback. The lesson will then be videoed and a copy given to the student for analysis, where after the student teacher will write a reflective essay on the practical implementation of the lesson.

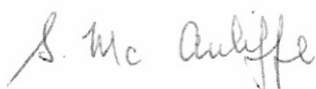
Based on the outcome of this investigation, we hope to expand this project to include in-service teachers, to provide training on Algebraic Thinking for teachers, and to develop materials for class work use that promotes learners' algebraic thinking.

The purpose of this letter to you is to seek your permission to allow us and your staff member to work together on this research. We have the permission of WCED's Research Directorate to conduct this research. Should a letter for parent consent be required, we will be happy to supply such a letter.

Therefore, we kindly request that you complete and return by fax (0866 479 420) the reply slip on the next page as soon as is convenient to you.

If you have any questions please feel free to contact me on (021) 680 1528 (W).

Sincerely



Ms S. Mc Auliffe
Senior Lecturer in Mathematics Education
Faculty of Education and Social Sciences
Cape Peninsula University of Technology

Can you indicate below (please mark with an X) how you would like to participate:

Name of school:

.....

Name of Principal:.....

1. I am prepared to allow the student teacher to teach a lesson which focuses on algebraic thinking	Yes	No
2. I am prepared to allow a researcher from the project to video-record the lesson	Yes	No

Name of Tutor teacher:.....

1. I am prepared to allow the student teacher to teach a lesson which focuses on algebraic thinking	Yes	No
2. I am prepared to allow a researcher from the project to video-record the lesson	Yes	No

- I understand that my school can withdraw from the research project at any time. I understand that I, my school, my teachers and my learners will remain anonymous outside the project.

 Yes **No**

- I want the student teacher and the CPUT research team to adhere to the following requirements (if any):

.....
.....
.....
.....
.....
.....

Please fax the reply slip to us at 0866 479 420 as soon as is convenient



Dear Parents

Please read the attached letter and indicate whether or not you give permission for your child to be part of a mathematics lesson that will be filmed for educational research purposes.

* I hereby give permission for _____ to be part of the maths lesson to be filmed for research purposes

* I hereby **do not** give permission for _____ to be part of the maths lesson to be filmed for research purposes
(there are boxes next to those options to tick)

Parent signature: _____

APPENDIX D

LESSON REFLECTIONS – CODING EXAMPLES FOR SCK (QUESTIONS)

Report: 24 quotation(s) for 1 code

HU: Data-Lesson Reflection_Version 2

File: [C:\data\PHD\ATLASFiles\LessonReflections\Data-Lesson Reflection_Version 2.hpr6]

Edited by: Super

Date/Time: 2012-07-05 17:21:04

Questions-Productive

P 5: S5-Video Reflection - 5:2 [My questions could have been m..] (6:8) (Super)

Codes: [Questions-Productive]

No memos

My questions could have been more clear, should have had more questions beforehand. I guided the learners too much. I didn't give them enough room to think on their own, to explore with maths.

P 6: S6-Video Reflection - 6:4 [I felt that my questions weren..] (10:10) (Super)

Codes: [Questions-Productive]

No memos

I felt that my questions weren't well thought through so the learners battled to understand what I was asking at times. This forced me to walk through every step with them with me doing a lot of the thinking for them rather than them understanding it themselves. I found myself asking "do you understand?" rather than me asking "what is your understanding?"

P 6: S6-Video Reflection - 6:6 [Like I said earlier I feel as ..] (16:16) (Super)

Codes: [Questions-Productive]

No memos

Like I said earlier I feel as if I could have improved my questioning at times. A lot of them were spontaneous so I didn't think through them before asking them.

P 7: S7-Video Reflection - 7:2 [One area I could have improved..] (5:5) (Super)

Codes: [Questions-Productive]

No memos

One area I could have improved was to guide the learners with lots of questions rather than me spoon feeding them and guiding them a long while I am doing it. The learners need to make the different steps and I need to guide them and make sure

they staying on track.

P 8: S10-Video Reflection - 8:1 [I feel maybe I did not ask the..] (11:11) (Super)

Codes: [Questions-Productive]

No memos

I feel maybe I did not ask the correct questions or questions that I could have got more information from the learners. I felt I was guiding the learners to much information. I wanted to feed off the learners and let them recognize and explore everything first for themselves which they did at times.

P 8: S10-Video Reflection - 8:10 [There where lots of questions ..] (25:25) (Super)

Codes: [Questions-Productive]

No memos

There where lots of questions asked (not always the right questions I wanted).

P 9: S8-Video Reflection - 9:1 [I think that the questions use..] (7:7) (Super)

Codes: [Questions-Productive]

No memos

I think that the questions used in the lesson were good and related to what was done in the lesson such as asking them to extend the picture and to add the values on a table, in order for them to arrive at a formula and also to plot the values on a graph.

P 9: S8-Video Reflection- 9:3 [I must admit that in terms of ..] (10:10) (Super)

Codes: [Questions-Productive]

No memos

I must admit that in terms of my questioning and the fast-approaching date for the video recording, I wondered if my learners would be able to think for themselves and actually answer and respond to questions. In a sense I also think that I partially underestimated them thinking that they would not have the ability considering that their Mathematics education journey has not been a good one.

P 9: S8-Video Reflection - 9:4 [I thought that asking them to ..] (12:12) (Super)

Codes: [Questions-Productive]

No memos

I thought that asking them to relate the formula to the picture would be tough for them but instead, they got the answer in about less than 10 seconds! I was totally amazed because even though I knew the answer and time was tight, I REFUSED to give them the answer.

P 9: S8-Video Reflection - 9:5 [The better questions in this I..] (14:14) (Super)
Codes: [Questions-Productive]
No memos

The better questions in this lesson were probably after they did the activity and we plotted the hexagon graph along with the square graph (on the same graph). Learners were able to see the formulas 'in play'. I could ask questions like why they think the hexagon was steeper than the square. I also added in some questions and asked the learners what quadrilateral(four sided) would have a similar growth compared to the hexagon and rectangle.

P10: S9-Video Reflection - 10:2 [When the learners responded to..] (15:15)
(Super)
Codes: [Questions-Productive]
No memos

When the learners responded to the questions that I posed, it gave me great pleasure to see that they actually understood what was being done and that they could come up with answers that I did not even notice.

P12: S13-Video Reflection - 12:4 [I should also be more aware of..] (24:24)
(Super)
Codes: [Questions-Productive]
No memos

I should also be more aware of the questions I ask and the answers I receive, and show more interest in the learners responses and ask "how they get their answers, so as to get some insight in where the learners are and if they are making the connections and discovering patterns.

P14: S15-Video Reflection - 14:3 [I feel that I had set out and ..] (18:18) (Super)
Codes: [Questions-Productive]
No memos

I feel that I had set out and asked a variety of questions in numerous ways that were at the level of the learners which helped extend their thinking and knowledge. I liked the fact that there was a mixture of open and closed questions as well as ones which got the learners to predict answers. I especially liked the question that I added during the lesson about what was happening to the piece of wool because this allowed the Grade 2's to see that although they were cutting their piece of wool and it was getting smaller, that it was still a whole but just now made up of parts. I think that my questioning covered a nice range of Mathematical concepts and not just the ones I specified for assessment in my lesson plan.

P14: S15-Video Reflection 14:9 [as I noticed that at times the..] (29:29) (Super)
Codes: [Questions-Productive]
No memos

as I noticed that at times the way I phrased certain questions left the Grade 2's confused. I feel that this could have been prevented if I have put myself into the Grade 2's mindset before the time and seen how they would have reasoned and by using simplified language in certain questions.

P14: S15-Video Reflection - 14:12 [I think that this was due to t..] (33:33) (Super)

Codes: [Questions-Productive]

No memos

I think that this was due to the fact that I was trying to cover too much during the lesson and because some of my questions asked weren't phrased correctly to get the responses I required. I do however feel that the more I practice this aspect the better I will get.

P15: S17-Video reflection - 15:5 [I think the fluidity and quest..] (10:10) (Super)

Codes: [Questions-Productive]

No memos

I think the fluidity and questions and extending questions and thinking will only come with more experience.

P16: S18-Video Reflection - 16:1 [The other thing I found diffic..] (4:4) (Super)

Codes: [Questions-Productive]

No memos

The other thing I found difficult is if a learner says or does something I haven't expected or planned for, I can't always think of a question as to why it has happened

P18: S20-Video Reflection - 18:2 [More clear questioning needs t..] (9:10) (Super)

Codes: [Questions-Productive]

No memos

More clear questioning needs to occur. Lots of why to help extend on learners thinking.

Need to find a variation of asking the same question so all learners will understand and the concept needs to be reinforced.

P21: S23-Video Reflection - 21:1 [I felt that I had good questio..] (8:8) (Super)

Codes: [Questions-Productive]

No memos

I felt that I had good questioning techniques, although I did miss out on a few learning opportunities when learners gave me wrong answers and I did not question

further to find out where their answers came from.

P22: S24-Video Reflection - 22:6 [With regards to paraphrasing t..] (20:20) (Super)

Codes: [Questions-Productive]

No memos

With regards to paraphrasing the learners I could have stimulated algebraic thinking by asking the learners questions like, “what do you mean when you say . . .”

P25: S12-Video reflection - 25:1 [the level of the questions, th..] (2:2) (Super)

Codes: [Questions-Productive]

No memos

the level of the questions, the clarity of the questions that were asked - here these questions needed to be clearer in what I was asking the students - I would present a question then get no response from the learners so would change the wording of the question once twice if not three times and this caused confusion and the lesson it be very bitty and up and down, the level of the question in terms of challenging the students more could have been improved.

P25: S12-Video reflection - 25:3 [so bringing asking them if we ..] (4:4) (Super)

Codes: [Questions-Productive]

No memos

so bringing asking them if we could not represent the shirts and pants and total outfits what would be another way that we could do this, and then the answer that you would looking for would be along the lines of using letters, symbols, pictures.

P26: S16-Video reflection - 26:3 [Asking meaningful questions th..] (5:5) (Super)

Codes: [Questions-Productive]

No memos

Asking meaningful questions that lead to higher order thinking was a challenge, something I feel I'd have to work out before the lesson because while the lesson is happening, the questions don't come naturally.

P26: S16-Video Reflection - 26:4 [I see how posing this problem ..] (6:6) (Super)

Codes: [Questions-Productive]

No memos

I see how posing this problem may have led to a bit of confusion rather than moving them on to catching the concept which they would have done if I'd started them off with a few variables.

VIDEO QUESTIONNAIRES – CODING EXAMPLES FOR SCK (TASKS)

Report: 17 quotation(s) for 1 code

HU: Video Questionnaire

File: [C:\data\PHD\ATLAS
Questionnaire.hpr6]

Files\Video

Questionnaire\Video

Edited by: Super

Date/Time: 2012-10-15 15:30:04

Selection and sequencing tasks

**P10: S1-Video questionnaire - 10:1 [This sequence was one which we..] (4:4)
(Super)**

Codes: [8. Selection and sequencing]

No memos

This sequence was one which we used in lectures at university when learning how to teach this concept. I found that people seem to understand and grasp the concept easier when following it in this sequence of activities, namely first looking at it represented in picture form then extending the pattern by adding the next picture(s), and/or representing it in a table, and then finding the function in either a generalized formula or flow diagram. Once doing this, they are able to eventually move onto representing it in a graph. This sequence I feel is a good way in further understanding algebraic knowledge and furthering algebraic thinking.

**P10: S3-Video questionnaire - 10:2 [I phoned other students and as..] (9:9)
(Super)**

Codes: [8. Selection and sequencing]

No memos

I phoned other students and asked them for help....this is basically what influenced my selection and sequencing of my activities. I did however choose the string activity, because I realised that it had potential to become a very concrete activity which would help the learners in grasping this brand new concept.

**P10: S4-Video questionnaire - 10:3 [The reason for bring in money ..] (12:12)
(Super)**

Codes: [8. Selection and sequencing]

No memos

The reason for bring in money was to show the learners that the concept of equal doesn't occur in the math classroom but outside as well. I asked the learner how much money must be given to a learner in order to ensure that the learners have equal amount of money.

P10: S6-Video questionnaire - 10:4 [I used activities which went f..] (18:18)
(Super)
Codes: [8. Selection and sequencing]
No memos

I used activities which went from easy to more difficult to test how much these grade sixes can deduce themselves.

P10: S7-Video questionnaire - 10:5 [So I thought lets tweak it a b..] (22:22)
(Super)
Codes: [8. Selection and sequencing]
No memos

So I thought lets tweak it a bit and make it more my lesson that why I added the spider grams and did the hexagon and not the pentagon. In the lesson I still touched on the pentagon asking the learners if they knew how many sides a pentagon had and what one thought of when told a pentagon? Lots of the learners said the Pentagon in America. I also explained that one gets a pentagon in baseball. This is really what influenced my selection for the lesson and sequencing of activities. I made the activities flow. So I did the triangle completed it with spider diagram and moved on.

P10: S8-Video questionnaire - 10:6 [Upon being at the school, I no..] (26:26)
(Super)
Codes: [8. Selection and sequencing]
No memos

Upon being at the school, I noticed that mathematics was taught in an abstract way. It was strictly a textbook teaching method. I therefore decided to first put the mathematics in context so that learners can see that mathematics is real and practical. I decided to give them a scenario and then to work from there to determine which tile would be a better option for Mrs Davies so that they felt that they were working towards a goal.

P10: S9-Video questionnaire - 10:7 [The fact that I had a grade fo..] (29:29)
(Super)
Codes: [8. Selection and sequencing]
No memos

The fact that I had a grade four class made me want to keep my lesson very interesting and relevant as well. The learners always enjoyed when animals, people or objects are used in lessons and for that reason, I chose the activity of the growing caterpillar. I started my lesson with a few learners standing in front of the class. This got everyone's attention

P10: S10-Video questionnaire - 10:8 [So the learners level and skil..] (37:37)
(Super)

Codes: [8. Selection and sequencing]

No memos

So the learners level and skills influenced my choice of activity's and I didn't know there algebraic skills so well. Needed a fun introduction and hands on activity throughout lesson.

**P10: S12-Video questionnaire - 10:9 [The sequencing of the activity..] (44:44)
(Super)**

Codes: [8. Selection and sequencing]

No memos

The sequencing of the activity was influenced by the progression of the lesson and to get the students to the possession of being able to generalize and create formulas or rules.

**P10: S13-Video questionnaire - 10:10 [I was under the impression tha..] (47:47)
(Super)**

Codes: [8. Selection and sequencing]

No memos

I was under the impression that I was to cover the whole process with the learners but after reviewing my lesson I realise it makes more sense to go at the learners own pace, allowing them to gather new information gradually and scaffolding their knowledge one step at a time. It would be far more beneficial for the learners to have full understanding of each step in the process before diving into finding a rule for which they have no understanding why we are doing so.

**P10: S15-Video questionnaire - 10:11 [I wanted to start the lesson o..] (63:63)
(Super)**

Codes: [8. Selection and sequencing]

No memos

I wanted to start the lesson off with what was known to the learners i.e. what was familiar to them. We started with counting in 2's as well as problems to do with 2 more or 2 less, doubling and halving.

**P10: S17-Video questionnaire - 10:12 [I started on the mat, with the..] (69:69)
(Super)**

Codes: [8. Selection and sequencing]

No memos

I started on the mat, with the whole class, and me in charge, showing the learners the outfits using actual clothes, because this was a very concrete method. It was also non-threatening as the class was together as a whole, and they were dependent on me. At this stage I was feeding them the information. I then had them go back to their desks. And take out their books. This moved them to a more independent stage

where they would have to draw from what they had learned to find their own solutions. They were also moving away from the concrete by having to draw their outfits in their books, and coming up with their own methods as well. Finally they become totally independent of me and had to use any means they knew or had learned in the lesson and together with what knowledge they had already acquired throughout the lesson they had to find their own solution. I chose the outfits activity because I thought it was a fun way to introduce pattern in a way they had not seen. It linked algebra to real life as well as allowing a fun way to come up with solutions (dressing up, and drawing pictures). I also used the table to record our data to show the learners one way to organise their findings, as well as make it easier for the learners to find the pattern that was developed through the increase of pants to the amount of outfits that could be made.

**P10: S20-Video questionnaire - 10:13 [I began with questioning learn..] (81:81)
(Super)**

Codes: [8. Selection and sequencing]

No memos

I began with questioning learners on their prior knowledge. I needed to find out what the learners knew and extend on that. I thought I help them make connections from knowledge known to the unknown knowledge. It helps them build up their knowledge progressively.

**P10: S21-Video questionnaire - 10:14 [In order for me to capture the..] (84:84)
(Super)**

Codes: [8. Selection and sequencing]

No memos

In order for me to capture their attention was to use the colour pictures and add the shopping story about twins to get their attention to the lesson. By showing them how things are done in reality and linking the concept to a situation where they themselves could be in then I thought that maybe they would understand better when they had to do the worksheet.

**P10: S22-Video questionnaire - 10:15 [My influence therefore, was to..] (91:91)
(Super)**

Codes: [8. Selection and sequencing]

No memos

My influence therefore, was to teach all the concepts they needed before the activity so that they didn't struggle during the activity.

**P10: S25-Video questionnaire - 10:16 [I used the basic steps we had ..]
(100:100) (Super)**

Codes: [8. Selection and sequencing]

No memos

I used the basic steps we had used in class, but I did add in the extra step of using a flow / spider diagram because the learners would not have been able to develop a formula / rule from the table. I also made sure the steps were not too much for the learners, giving them time in between to fill in tables and charts, allowing them to catch their breaths.

**P10: S26-Video questionnaire - 10:17 [Initially I liked the problem ..] (103:103)
(Super)**

Codes: [8. Selection and sequencing]

No memos

Initially I liked the problem of working out the “Dogs eyes” and I thought that adapting it to fit the school’s theme seemed rather simple, since there are several farm animals with the same basic features as the original problem, i.e.: 2 eyes, 2 ears, 1 nose, 1tail, 4 legs ect... also it worked with a basic concept of either group/skip counting and/or doubling/halving. Also there was room to add an extra *problem* in the form of the 2 eyes and 1 tail scenario, which I should have included, but again I did not anticipate their being able to complete and solve the problem as easily as they did.

FOCUS GROUP INTERVIEW – CODING EXAMPLES FOR SPECIALISED KNOWLEDGE FOR TEACHING (SCK)

Report: 30 quotation(s)

HU Data-FG Interviews-V2

File: [C:\data\PHD\ATLAS Files\Data-FG Interviews-V2.hpr6]

Edited by: Super

Date/Time: 2011-10-26 22:34:34

Code – Specialised Content Knowledge - SCK

P128: FG7 - Final Interview ISP.doc - 128:4 [I just feel that it is more pr..] (19:19) (Super)

I just feel that it is more practical to me because when I look at it, I can relate it immediately to the world and to the things that we actually don't do. How the children must survive like that whereas at school I just learn things as procedures or structure. This is that or that formula but I don't know why I did it whereas now I know, o, I am not only doing that for the fun of it but I can actually use it in my everyday life one day.

- Selecting representations for different purposes (everyday life usage)

P128: FG7 - Final Interview ISP.doc - 128:8 [For me the question is, in tea..] (27:27) (Super)

For me the question is, in teaching practice when you teach maths, children always ask you why must I do this. I'm not going to use it everyday but when you use it in real life situations I say you are going to use it and this is where you are going to use it and this is how you going to...

- Selecting representations for different purposes (everyday life usage)

P128: FG7 - Final Interview ISP.doc - 128:17 [One the thing that I definitel..] (57:57) (Super)

One the thing that I definitely found was that they really enjoyed linking it to real life situation and I actually had kids come up to me afterwards and say like thank you mam for linking it to real life and o, I actually understand now.

- Selecting representations for different purposes (everyday life usage)

P128: FG7 - Final Interview ISP.doc - 128:19 [algebra or in any maths lesson..] (66:66) (Super)

algebra or in any maths lesson especially my questioning techniques needs... somehow I need some help in doing that because the learners, its not that they didn't know but I didn't know how to ask them the right questions for them to give me the right answer.

- Asking productive questions (problem)

P128: FG7 - Final Interview ISP.doc - 128:20 [You want to go on with your le..] (69:69) (Super)

You want to go on with your lesson so you want them to give you the knowledge, you keep them on the knowledge questions about what do you know. But you don't want to go higher so that they can think.

- Asking productive questions (problem)

P128: FG7 - Final Interview ISP.doc - 128:21 [My class, they were weird thou..] (72:72) (Super)

My class, they were weird though because some were capable of doing the maths and others had the maths anxiety behind it so they knew it but didn't want to contribute to the class but then when it was put forward in a different light, when used in different examples and stretched them a bit that I hadn't done was with x, y and z. They then yes got it so also it is a bit of Zita and Dannie's.

- Make and use mathematical representations effectively (presenting problems in a different way)

P128: FG7 - Final Interview ISP.doc - 128:23 [Like Megan said, she spoke abo..] (78:78) (Super)

Like Megan said, she spoke about don't underestimate the class. In my class I had extremely weak learners. Don't underestimate a certain learner who you think is weak because that was the learner that knocked me off my socks that day.

- Selecting examples with pedagogically strategic intent

P128: FG7 - Final Interview ISP.doc - 128:25 [so when you were like doing di..] (90:90) (Super)

so when you were like doing different ways of explaining I was like just basically learning them like new, not like I've done it already in matric. I only knew of one way and that was it and all this was very new to me and the catch up was like what, what, what. I was just basically grabbing it for the first time. So trying to keep up was quite difficult.

- Presenting mathematical ideas (different ways of explaining)

P128: FG7 - Final Interview ISP.doc - 128:29 [Because we need to understand ..] (98:98) (Super)

Because we need to understand what you do. We make sense of it that way. So that you can see there is many different like answers just for one thing and then one answer you may find actually easier to do, the one might be easier than the one that

you made wrote learn into us. Somebody else's method might be o, okay I understand your way. I'm going to use your way and somebody else might say I prefer this way because it's a lot more..

- Understanding different interpretations of mathematical solutions

P129: FG8 - Final Interview FP.doc - 129:12 [I think, I don't actually know..] (62:62) (Super)

I think, I don't actually know what I'm thinking but I think the way that I put it towards them could have been scaffolded a little bit more than what I did. I think the lesson that I did with them was up until a certain level but I didn't know how to take it further.

- Asking productive questions

P129: FG8 - Final Interview FP.doc - 129:13 [I didn't know that's what I'm ..] (65:65) (Super)

I didn't know that's what I'm still trying to think exactly what to say, like I didn't know...if they said something I didn't know how to take it and say okay so you say this, so is she right, is she wrong, question her because I was at a girls school. Like question her. The questioning techniques and the modelling techniques. I was very inexperienced.

- Asking productive questions

P129: FG8 - Final Interview FP.doc - 129:14 [Because sometimes they come up..] (65:65) (Super)

Because sometimes they come up with this like profound statements like "how did you get" and then you think oh okay. Like how do you break it down so the rest of them can understand it? How do you do that and how do you?? them because now they already five steps ahead of everyone else and you have to kind of push them but where to.

- Presenting mathematical ideas

P129: FG8 - Final Interview FP.doc - 129:15 [I think using concrete like th..] (71:71) (Super)

I think using concrete like that is a lot easier for them to understand but then if I would do it again I would break it down a little bit more. Something like that because they got the general idea but then they ended up copying each other.

- Presenting mathematical ideas

P129: FG8 - Final Interview FP.doc - 129:17 [We had the theme of farm anima..] (82:82) (Super)

We had the theme of farm animals so I kind of just went with that and I did the whole

one cow and how many eyes. So they already kind of already knew what was going on and I had a lot of them play with number, they had to complete, they had their workbooks, they had their pens and pencils so they could write down everything or make up their own little things as they were going along. A lot of them surprised me like I didn't think I would get to the part where you could add a table, like you know when you say "I have one cow, how many eyes and tails would you have". They had progressed from the eyes to adding a tail almost effortlessly without me having to do anything or say anything. This one little boy what he did was he filled out his table and he'd write two numbers, so he would write the number of eyes and the number of tails so it looked like he wrote a two digit number and then in the middle he'd write the total so it kind of added the two together but he didn't quite get that he only needed to write one number and when he explains it he was like actually I could've just counted in 3's. So he sat there and he was working it all out and as he was explaining it to me he stops middle sentence and he says like mam I can just count in 3's because I know I count in 3's everyday.

- Selecting examples with pedagogical intent

P129: FG8 - Final Interview FP.doc - 129:18 [You know and I think what we d..] (82:82) (Super)

You know and I think what we do is that we kind of just give them something and they do it but you don't expect that they understand at that level that we do. They understand it a little bit differently than we do but you know they do get it. We underestimate them in a way and I know I did when I gave them the cows and the eyes.

- Modifying tasks to make easier or more difficult

P129: FG8 - Final Interview FP.doc - 129:19 [I taught a couple more Maths I..] (86:86) (Super)

I taught a couple more Maths lessons and we did a few more like eyes and ears and tails and legs and blaa blaa blaa. And challenging them that little bit more and asking it in a different way, things came across so much more easily and its not like they understood it, its just that they were eager to try something different because when they see a plus sign its like automatically that but when you give it to them in a different way or you put it in a word sum they've got that function way, that procedure of this is what needs to be done but if they have to think about the how and the why, they get all excited and you know they want to do more.

- Selecting examples with pedagogical intent

P129: FG8 - Final Interview FP.doc - 129:27 [Like Kerri said in high school..] (130:130) (Super)

Like Kerri said in high school if I had gotten that question I would never ever in a million years have thought to think like okay I have this that I know I need but I also know I need this used in an equation must give me that answer. So let me try and

use the answer to try and get what I actually should have had at the beginning. I would never have thought about it like that.

- Connecting topics to prior and future learning

P129: FG8 - Final Interview FP.doc - 129:28 [Yesterday I was tutoring physi..] (132:132) (Super)

Yesterday I was tutoring physics and they were trying to get current or something and there was like a million equations to get current. I mean all of it has current in it so they were like but I've got all this information now how do I get current. Like see which information you've got and which one you need and they were like okay but I can use this equation or I could use this one or I could use this one because they could see the relationship between the different aspects of the information that they had.

- Connecting topics to prior and future learning

P129: FG8 - Final Interview FP.doc - 129:30 [And we broke it up to why and ..] (148:148) (Super)

And we broke it up to why and how we got this formula. How you could link it to the other side whereas in high school it was more this is how you do it. And A is for that and B is for that.

- Explain and justify mathematical ideas

P129: FG8 - Final Interview FP.doc - 129:31 [It's important to teach them t..] (151:151) (Super)

It's important to teach them the "Why" and "How" you got to that so that they can understand the concept and they can remember it.

- Explain and justify mathematical ideas

P129: FG8 - Final Interview FP.doc - 129:32 [Don't just look at it from one..] (151:151) (Super)

Don't just look at it from one way, a problem has a whole lot of entrances, you can tackle a problem from anywhere and its things like that so when I teach one day I want to explain the "why and the how" but it's not something they have to learn off by heart and that's I think...

- Explain and justify mathematical ideas

P129: FG8 - Final Interview FP.doc - 129:33 [I knew you were tackling a pro..] (151:151) (Super)

I knew you were tackling a problem from a different way and I knew that it worked. So it was almost like it gave us the opportunity to see you can use different ways

- Understanding different interpretations of mathematical solutions

P129: FG8 - Final Interview FP.doc - 129:34 [When it comes to this, like yo..] (153:155) (Super)

When it comes to this, like you said you can come from different, you can go to it from any angle and figure it out and do it backwards. And you don't have to learn it off by heart.

- Explain and justify mathematical ideas

P129: FG8 - Final Interview FP.doc - 129:35 [Because you can figure it out ..] (159:161) (Super)

Because you can figure it out because of what you know and more importantly what you understand. You basically use your own logic to get to your own conclusion to get to the same thing.

- Explain and justify mathematical ideas

P129: FG8 - Final Interview FP.doc - 129:44 [I mean when I teach maths I do..] (210:210) (Super)

I mean when I teach maths I don't want to teach in a way like I give you a formula and you spit out the answers. I want to know that you understand every little bit of that and the "how and the why" and I want to be able to not just teach it but teach around it.

- Explain and justify mathematical ideas

P129: FG8 - Final Interview FP.doc - 129:46 [I know we always say we set it..] (213:213) (Super)

I know we always say we set it in the context but you do because when you set it in a context, you do it and it makes sense and you remember it.

- Selecting representations for different purposes (everyday life usage)

P129: FG8 - Final Interview FP.doc - 129:48 [I remembered it because I reme..] (215:215) (Super)

I remembered it because I remembered the real life stuff because they like stuck in my head the most. So obviously when I teach I am also going to try and make it a lot of real life examples and show them how it relates to their everyday lives and things like that.

- Selecting representations for different purposes (everyday life usage)

**P129: FG8 - Final Interview FP.doc - 129:50 [That would stick with me becau..]
(217:217) (Super)**

That would stick with me because now I can look at a problem and go about it in different ways and not just one way because you can forget one way and if you forget that way you can go a different way.

- Understanding different interpretations of mathematical solutions

**P129: FG8 - Final Interview FP.doc - 129:53 [Like Yusra said what I'll take..]
(225:225) (Super)**

Like Yusra said what I'll take away from this is you need to link everything because you can't teach in isolation because then it will never ever make sense because in high school I never actually understood what Maths was all about because I've been taught that and that and this is the way how to do it and this is the way of doing that and now when I came here and it was like okay these were the ways you can do it and look at it from a different perspectives

- Connecting topics to prior and future learning