Cape Peninsula
University of Technology

THE DEVELOPMENT OF ALGEBRAIC THINKING IN THE FOUNDATION PHASE: A COMPARATIVE STUDY OF TWO DIFFERENT CURRICULA.

by

DOMINIQUE GABRIALA AFONSO

Thesis submitted in fulfilment of the requirements for the degree

Master of Education

in the Faculty of Education

at the Cape Peninsula University of Technology

Supervisor: Dr S Mc Auliffe

Mowbray

March 2019

DECLARATION

I, Dominique Gabriala Afonso, declare that the contents of this dissertation/thesis represent my own unaided work, and that the dissertation/thesis has not previously been submitted for academic examination towards any qualification. Furthermore, it represents my own opinions and not necessarily those of the Cape Peninsula University of Technology.

Signed **Date**

ABSTRACT

The mathematics results in South Africa are alarmingly low, with a number of high school learners unable to compute basic operations. International test results show South Africa consistently ranks low in comparison to other countries whilst Singapore continues to perform well. Some schools in South Africa have decided to adopt the Singaporean method of teaching mathematics, known as Singapore Maths, in the hope of improving learner results. This study seeks to understand how two different curricula, South African and Singapore, provide opportunity for the development of algebraic thinking in the Foundation Phase. There is ongoing research which suggests a link between algebraic thinking (Early Algebra) and a deeper conceptual understanding of mathematics (Blanton & Kaput, 2003).

This study comprises a qualitative case study of two schools using different curricula and textbooks to teach algebraic thinking with a special focus on patterns and functional thinking. Data were gathered using document analysis of curriculum and textbooks; learner tests; semi structured interviews with class teachers and focus group interviews with Grade 3 learners from each curriculum group. The analysis process involved pattern matching and building explanations related to each data collection instrument using Blanton, Brizuela, Gardiner, Sawrey and Newman-Owen's (2015) levels of sophistication in learner's thinking about functional relationships. The results of the study suggest that although South African learners have the potential to think algebraically, they are not, however, always offered the opportunities to do so. The importance of suitable mathematical activities and scaffolding is highlighted and the critical need for professional development for teachers in which the importance of Early Algebra is defined and explained. It is imperative that the curriculum and textbooks activities are relooked at to address the development of algebraic thinking in the early grades and shift the focus from an emphasis on arithmetic relationships to thinking in generalised ways about functional relationships.

ACKNOWLEDGEMENTS

My sincere and heartfelt thanks go out to the following people who have made this study possible:

- The financial assistance of the National Research Foundation towards this research is acknowledged. Opinions expressed in this thesis and the conclusions arrived at, are those of the author, and are not necessarily to be attributed to the National Research Foundation.
- To the principals, teachers and learners of the schools where this study was conducted. Thank you for your valuable contribution.
- To my colleagues at Edgemead Primary School. Your love, support and understanding mean the world to me…thank you.
- To my supervisor, Dr Sharon Mc Auliffe. Thank you is simply not enough. Your support and guidance throughout this study has been inspiring. Thank you for motivating me to be my best. I hope to inspire my learners the way you have inspired me.
- To my amazing family and friends who have loved and supported me through this process. To Aunty Colleen, Leandra, Natasha and Rosanne - I could not have done this without you. Thank you for all that you did and continue to do for me. I appreciate it more than you'll ever know.
- To Tyrone - my rock, my support and my number one cheerleader. Thank you for your love, your patience and for walking this journey with me, every step of the way.
- To my parents, Mario and Lidia Afonso. Thank you for your constant and unconditional love, care and support throughout this process. For the pep talks, the hugs, the motivational speeches and the countless tissues to dry my tears. Thank you for always believing in me…even when I didn't believe in myself.
- Thank you to God, through whom everything is possible.

DEDICATION

To my precious Juliana

For teaching me the true meaning of perseverance, dedication and hard work.
Thank you for inspiring me, each and every day.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

APPENDICES

CHAPTER 1

INTRODUCTION

1.1 Introduction and background to the study

Mathematics results in South Africa are generally poor, with a larger percentage of learners achieving far below the international norm. The Trends in International Mathematics and Science Studies (TIMSS) in 2015 show that South Africa's mathematics results are alarmingly low with many high school learners unable to perform basic computations and some being described as completely innumerate (Mullis, Martin, Foy & Hooper, 2016; Spaull, 2013).

Singapore is one of the countries that continues to lead the world in mathematics achievement and outperformed every country in the TIMSS 2015 study (Mullis *et al*., 2016). In 2011 and 2015, South Africa participated in the TIMSS study for the Grade 8 level and administered the tests to Grade 9 learners. Singapore was ranked first overall in the TIMSS mathematics tests in 2015, while South Africa placed second from the bottom out of 45 countries (Mullis *et al*., 2016) and appeared below the low-performance benchmark. These results appear to show that Singapore's mathematics curriculum is producing results and leads us to wonder if it would be possible for South Africa to do the same.

Singapore mathematics is based on the Singaporean system of mathematics and is centered around problem solving (Chen, 2008). It requires the learner to construct a deep understanding of each concept, progressing from concrete to abstract, in order to assist the learner on this trajectory. Singapore mathematics emphasises conceptual understanding and procedural fluency resulting in a deeper understanding of the concept being covered (Chen, 2008).

Algebra makes up a large portion of the high school mathematics curriculum. For many years, researchers have focused on algebra and the learners' understanding in this area (Vermeulen, 2008). Carpenter, Franke and Levi (2003) believe that many learners find algebra difficult, which results in poor mathematics performance later in their school careers. Their research suggests that if learners genuinely understand arithmetic and are able to explain and justify their calculations, they will have learnt some of the critical foundations of algebra. This type of mathematical thinking can provide a solid foundation for learning and understanding algebra. It must, however, be developed over time and should start in the Foundation Phase (Carpenter *et al*., 2003). Blanton and Kaput (2004) agree, and believe that children need to learn a way of doing mathematics that is different to their parents. They need to be taught mathematics that extends further than arithmetic and computational fluency and instead includes the opportunity for learners to build, express and justify mathematical generalisations. Vermeulen (2008) supports this approach and recommends that algebraic thinking (Early Algebra) should be developed in the lower grades before the learner is exposed to algebra in high school. Early Algebra is not about introducing algebra procedures and skills earlier in the primary school but rather about developing a way of thinking and reasoning that benefits all aspects of mathematics (Mc Auliffe, 2013).

The curricula used in this study have been specifically chosen in order to create a comparison between two systems which are currently available in different South African schools. For the purpose of this study, the term *comparison* is not used in the traditional sense. The study does not aim to determine which curriculum is best, but rather aims to *compare* them in terms of their similarities and differences regarding the development of algebraic thinking and the opportunities offered to do so. The first is the South African Curriculum and Policy Statement (CAPS) which is used in all public schools and some independent schools. The other curriculum in this study is the Singaporean mathematics (SM) curriculum, available in some South African schools, both public and independent. The aim of the study is to compare the South African CAPS and Singaporean mathematics curricula and textbooks in order to understand teachers' interpretations of each curriculum as well as monitor learners' performance. An understanding of how the different curricula provide opportunity for the development of algebraic thinking in the Foundation Phase can then be established.

1.2 Rationale

The Constitution of the Republic of South Africa recognises access to quality education as a basic human right. Although the national pass rate for the Senior National Certificate has seen an improvement from 2016, South African learners' performance in national and international studies does not illustrate that of quality instruction (Arends, Winnaar & Mosimege, 2017). The TIMSS study shows South Africa performing far lower than the international mean, despite the test being administered to learners a year older than their international counterparts (Mullis *et al*., 2016). The South African national grade average for Grade 3 mathematics in the 2014 Annual National Assessments

(ANA's) was 56% and the Grade 9 average for mathematics dropped to an alarming 11% (SA.DBE, 2014). There have been some improvements and while South Africa continues to perform in the lower end of the ranking scales, the country has shown the biggest improvement of the 25 countries who have participated in both the TIMSS 2003 and 2015 Grade 9 study (Reddy, Visser, Winnaar, Arends, Juan, Prinsloo & Isdale, 2016). Singapore remains one of the highest ranking countries for both mathematics and science in the TIMSS and the Programme for International Student Assessment (PISA). A number of South African schools, both public and private, have started to teach mathematics using the Singapore mathematics approach in an attempt to improve the performance of learners. Mc Auliffe and Lubben (2013) suggest that knowledge and application of algebra makes up a significant portion of Grade 12 mathematics. They argue poor mathematics matric results show that learners have not developed the necessary knowledge and skills and propose that a different approach needs to be taken in the teaching of algebra to improve mathematical understanding.

In order for South Africa to improve its mathematics results, a change in the way children learn and understand mathematics needs to take place. Research has shown a significant link between algebraic thinking (Early Algebra) and a more conceptual understanding of mathematics. Algebraic thinking is described by Blanton (2008:6) as the building, expressing and justifying of mathematical relationships and is designed to help children to "see and describe mathematical structures and relationships for which they can construct meaning". This way of thinking about and engaging in mathematics allows the learner to construct conceptual understanding of the concept being covered. The role of the curriculum is important as it guides the teacher in providing opportunities for these expressions and justifications to take place. The rationale behind this study was to ascertain how two mathematics curricula, currently used in South African primary schools, provided for the development of these Early Algebraic skills to better equip learners for future mathematics.

1.3 Purpose

This research was prompted by the alarmingly low mathematics results as seen in assessments such as the ANAs and the TIMSS. Research relating to Early Algebra states how an understanding of basic algebraic principles is required in order to possess a deep, conceptual understanding of mathematics (Carraher, Martinez & Schliemann, 2008). Cai and Knuth (2005) propose that learners should be introduced to these algebraic skills from an early ages, as they have the potential to do so (Gotze, 2016). By answering the research question and sub-questions that follow, this study aims to understand how two mathematics curricula used in South African primary schools assist in developing algebraic thinking. This was researched by consulting curriculum documents and textbooks, interviewing teachers as well as testing learners from two different teaching contexts. The learners completed a pencil and paper test and some were selected to participate in focus group interviews.

1.3.1 Research title

The development of algebraic thinking in the Foundation Phase: a comparative study of two different curricula.

1.3.2 Research question

The main question and sub-questions which guided this study were as follows:

How do the South African and Singapore mathematics curricula provide for the development of algebraic thinking in the Foundation Phase?

Sub-questions:

How is algebraic thinking developed in the South African and Singapore mathematics curricula and textbooks for Foundation Phase?

What are the teachers' perceptions of the development of algebraic thinking within each curriculum?

How do learners using CAPS and SM curricula solve pattern problems?

1.4 Literature overview

The literature review provides an overview of current research related to three different aspects of this study: Early Algebra, curriculum and textbooks. The last section of the chapter deals with the theoretical framework used to inform the research and to analyse the data. Early Algebra, otherwise known as algebraic thinking, is a relatively new field of study in mathematics education and there is still no agreed definition on what it means. Different definitions are presented and a definition of algebraic thinking, as used for the purpose of this study, is explained and justified. There is also some discussion of the different aspects of Early Algebra, namely generalised arithmetic and functional thinking. This is followed by a focus on studies related to curricula and algebraic thinking with reference to South Africa and Singapore. Textbooks play a crucial role in the mathematics classroom as they are often the translation of the curricula for the purpose of teaching and learning (Garner, 1992; Haggarty & Pepin, 2002). This is explored in more detail in the context of Early Algebra. The theoretical framework section discusses two different frameworks to analyse learners' thinking when solving functional thinking type problems. These were developed by Twohill (2013) and Blanton, Brizuela, Gardiner, Sawrey and Newman-Owens' (2015). The frameworks were compared to discern the links between them and to ascertain which would be most suitable for the purpose of this study. The levels of sophistication of children's functional thinking, developed by Blanton *et al*. (2015), was adopted as the principle theoretical and analytical framework.

1.5 Overview of methodology

This research study is qualitative in nature and makes use of a case study approach. The site of the research was two schools, each following a different mathematics curriculum. A Grade 3 class from each curriculum group, along with the class teachers, made up the sample for the study. Data were collected using different collection methods. A pencil and paper test was administered to each learner by the class teacher and the teachers were interviewed using semi-structured interviews. A mixed ability group of eight learners per class was selected to be part of two focus group interviews. These focus group interviews were audio-recorded while the learners discussed and solved functional thinking type pattern problems.

Frameworks developed by Blanton *et al*. (2015) and Tarr, Reys, Barker and Billstein (2006) were used as analytical tools for the data analysis process. The curriculum documents and textbooks went through a two-step analysis. They were first analysed using the Tarr *et al*. (2006) framework and then a second level of analysis was completed using the Blanton *et al*. (2015) framework. The pencil and paper scripts were graded and specific questions selected for further analysis using the Blanton *et al*. (2015) framework. The reasons for this selection are explained in Chapter Three. The teacher interviews and focus group interviews were transcribed and the responses and solution methods analysed by classifying them using the levels of sophistication (Blanton *et al*., 2015).

1.6 Significance of the study

Research has revealed that a deep understanding of algebraic thinking assists in the conceptual understanding of mathematics (Blanton, Brizuela, Stephens, Knuth, Isler, Gardiner, Stroud, Fonger & Stylianou, 2018). Gotze (2016) suggests that young learners' algebraic thinking skills need to be developed in the early grades through exposure to tasks that help develop these skills. The findings of this research can inform curriculum and textbook design, where the focus on pattern activities is shifted from arithmetic to algebraic thinking. This, along with teacher professional development, could increase the opportunities offered to the learners to think algebraically. Teacher development should include an awareness and understanding among teachers on the importance of the selection and sequencing of algebraic tasks to build coherence and connections in teachers' mathematical discourses in instruction (Venkat & Adler, 2012). A reconsideration of the curriculum guidelines, learner activities and teacher instruction would be beneficial to developing learners' algebraic knowledge and skills in preparation for the study of more advanced algebra.

1.7 Scope and limitations of the study

There are some limitations to this study and to case studies in general. A small sample as given in this study does not allow for generalisations across a larger community. An in depth description of the site and sample, as specified in Chapter Three, does however allow for a deeper analysis of the issues related to the development of algebraic thinking and supports transferability and validity of the results to different contexts.

The role of the researcher can be seen as a limitation as one can bring an aspect of bias to the study (Olsen, 2012). Creswell (2014) states that the researcher should clarify the possible bias added to the study, which could be shaped by one's background, experience and culture. A description of this possible bias is outlined in Chapter Three where the researcher's experience teaching in the younger grades (Foundation Phase) and how this could affect the research is discussed.

1.8 Organisation of the study

Chapter One introduces the study and offers the background, rational, purpose and research problem. The focus the study is explained, where the aim is to understand how two different curricula provide for the development of algebraic thinking in the Foundation

Phase. The research topic and sub-questions are provided with an overview of the methodology and significance of the study.

Chapter Two outlines the literature and theoretical framework related to the research topic. The chapter begins with a review of Early Algebra literature and research. Curricula is defined and the two curricula used in this study, CAPS and SM, are explained. This chapter then provides insight into textbooks and how they are used within the context of the Foundation Phase. The chapter concludes with a description of the theoretical framework. Blanton *et al*. (2015) developed levels of sophistication which were used to analyse children's algebraic thinking when asked to generate and organise data for a given scenario and then to develop a function rule. This framework serves as both the theoretical and analytical framework.

Chapter Three describes the research design and methodology. It starts with an explanation of the design and methodology of the study, followed by a detailed discussion of the research plan. The site and sample are introduced and the data collection methods are described. The details of the data analysis process and the ethical considerations are further elaborated.

Chapter Four contains the results collected from the document analysis, learner tests, semi-structured and focus group interviews. These are discussed in relation to each sub-question and the overall development of algebraic thinking in the Foundation Phase.

The final chapter includes a summary and discussion of the results in relation to each of the sub-questions. Important issues are highlighted and correlated to the literature review and theoretical framework in Chapter Two. The chapter concludes with a reflection on the research question and possible recommendations for further research.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This chapter provides a detailed review of the literature pertaining to this study, including relevant theory. Fink (2014) explains that the aim of a literature review is to provide the reader with a vast understanding of the field of study in relation to the problem being researched. The aim of this study is to understand how two different mathematics curricula used in South African primary schools provide for the development of algebraic thinking in the Foundation Phase.

The chapter begins with a detailed description of literature related to Early Algebra, curricula and textbooks. Early Algebra has been the topic of many research studies which suggest the introduction of algebraic thinking skills should occur in the early years of formal schooling (Brit & Irwin, 2011; Gotze, 2016). The term algebraic thinking, as used for the purpose of study, is defined and explained. Curricula play an important role in this research as two different mathematics curricula are compared. A description of the structure of each curriculum is offered and the content related to patterns and pattern activities are explored in more detail. These activities provide the opportunities for the development of algebraic thinking. Furthermore, textbooks as an important resource in the mathematics classroom and effect on teaching and learning is explored in some detail (Garner, 1992). Literature related to theoretical frameworks that have been developed to analyse textbooks is also discussed. The chapter concludes with a comparison of two different theoretical frameworks which are used to analyse the level of children’s understanding of functional thinking.

2.2 Early Algebra

Algebra readiness has become an important part of the mathematics research field due to the poor performance of high school learners in algebra. Cai and Knuth (2005) believe that in order to alleviate this problem, young children should be better prepared to learn algebra. In order to do this, they suggest young school children be exposed to experiences that will assist them to think algebraically. Gotze (2016) agrees and states that children’s abilities to think algebraically should be developed in early grades as this will assist in conceptualising the deeper, underlying structure of mathematics. Extensive research on algebraic thinking in primary schools has already been undertaken and encouraging results have been produced. Blanton and Kaput (2003) believe that the development of algebraic thinking can improve primary school mathematics learning as it provides opportunities for the learners to develop a deeper conceptual understanding of mathematics. Therefore, if begun at the primary school level, it is believed the deep understanding learners develop will assist them with understanding further mathematical content. This mathematical way of thinking can provide a solid foundation for learning and understanding higher algebra. It must, however, be developed over time and should therefore start in the early grades (Carpenter, Franke & Levi, 2003).

2.2.1 Early Algebra: Definition

Algebraic thinking, otherwise known as Early Algebra, is described by Blanton (2008) as the building, expressing and justifying of mathematical relationships, allowing children to construct meaning of mathematical structures and relationships. Mc Auliffe and Lubben (2013) define Early Algebra as a way of teaching algebra where children are able to think and reason algebraically. Carraher, Schliemann, Brizuela and Earnest (2006) support this and further define Early Algebra as a move away from numbers and measurements towards seeing the relationships between sets of numbers and measurements. Mulligan, Cavanagh and Keanan-Brown (2012) explain that Early Algebra is not only thinking about algebra from an earlier age but includes re-looking at numbers from a different perspective.

Initial research in the early 2000's found problems with the arithmetic way of teaching and learning that was taking place in elementary classrooms. The learners were unable to move from an arithmetic to algebraic way of thinking when introduced to formal algebra in high school. Researchers then began to explore whether different types of algebraic activities introduced to younger learners could assist in the transition to formal algebra. As the children work through the problems, they discover relationships, draw conclusions and express generalisations, and then build arguments to support them (Blanton & Kaput, 2003). Cai and Knuth (2005) suggest that Early Algebra is not simply adding a simplified version of traditional algebra in the early grades. Instead, it is the development of a different way of thinking which includes analysing relationships between quantities, generalising, problem solving, modelling, justifying, predicting and proving (Kieran, 2004). Kieran, Pang, Schifter and Ng (2016) suggest that mathematical relations, patterns, and arithmetical structures lie at the heart of early algebraic development. In order to successfully engage learners, processes such as generalising, noticing, representing and communicating must be involved. Using natural language to express these generalisations is considered a fundamental part of the process. Carraher *et al*. (2008) maintains that in order to have a deep, conceptual understanding of mathematics, the generalisation and understanding of basic algebraic principles is required.

Blanton *et al*. (2018:37) state that "Early Algebra has the potential to embed arithmetic concepts in rich algebraic tasks in ways that can deepen children's understanding of arithmetic concepts". They explain that by including Early Algebra as part of the current mathematics happening in the classroom, a deeper understanding of mathematical knowledge may be built (Blanton *et al*., 2018). Mc Auliffe (2013) emphasises the importance of providing opportunities for children to develop their algebraic thinking in the early grades and for this to continue into high school. If these opportunities and activities are carried out in a meaningful way, Carraher *et al*. (2006) believe it has the potential to bring together mathematical ideas that may have otherwise remained isolated. The design and use of mathematical tasks are important factors when developing algebraic thinking. When using existing tasks, teachers need to find the opportunities within the task to develop the child's algebraic thinking, or need to redesign the task if these opportunities do not exist. Teachers need to be aware of the spontaneous opportunities that arise in the classroom to encourage the learners to think algebraically and must make use of them to develop this way of thinking (Hunter, 2016).

Greenes (2004) identifies three key aspects (big ideas) involved in the instruction of Early Algebra, namely variables and equations, generalised properties, patterns and functions, and highlights the critical role of the teacher. Variables are letters or shapes which are used to represent unknowns and quantities and to make generalisations of properties. The unknown quantity being represented could have a fixed value or a varying value. When children are exposed to the relationships between numbers using specific operations, they indirectly apply algebraic rules such as the commutative property of addition and the identity property of multiplication. Children are generally exposed to two types of patterns in the younger grades - repetitive patterns and growing or shrinking patterns. The latter leads the child to make generalisations and later to represent these using variables. The children are encouraged to make use of function tables to find the pattern and record the rule. The role of the mathematics teacher is vital in helping to make explicit connections between the algebra and arithmetic. The teacher needs to select appropriate tasks which allow children to spend time solving problems in different ways and provide guidance in explaining their thinking.

2.2.2 Early Algebra: generalised arithmetic and functional thinking

Most Early Algebra research focuses on two aspects of algebra: namely *generalised arithmetic* and *functional thinking*. Blanton (2008) defines generalised arithmetic as the use of mathematics to produce generalised mathematical statements about the operations and properties of numbers. Blanton further explains that these generalisations may start with words, but as the child's mathematical knowledge and vocabulary mature, these generalisations will be expressed more symbolically. Through this generalised arithmetic, children are able to generalise and learn important mathematical concepts such as commutativity and equality (Blanton & Kaput, 2003). In a recent study conducted by Blanton *et al*. (2015) with young children it was found that 75% of the Grade 3 participants were able to use a variable notation to represent an unknown quantity. Matthews, Rittle-Johnson, McEldoon and Taylor (2012) designed specific tests aimed at investigating learners' understanding of the equal sign. They found that learners who possessed a deeper understanding of the equal sign were able to solve more complex equations, therefore suggesting a direct link between the equal sign and algebraic thinking. Brit and Irwin (2011) developed a curriculum for the development of algebraic thinking and found that learners following it were more successful than those following an arithmetic based curriculum when solving test questions involving both simple addition and subtraction computations and more complex questions involving fractions. Another study with older learners aged 12-14, found that early and consistent exposure to algebraic thinking in younger grades results in learners being better equipped to make generalisations involving the alphanumeric symbols of algebra in later grades (Britt & Irwin, 2011).

The second component of Early Algebra is *functional thinking*. Mulligan, Cavanagh and Keanan-Brown (2012) define functional thinking as seeing expressions and equations as a whole, where the approach is not the same as the one used to solve the computation using a sequenced procedure. Blanton (2008) explains that functional thinking involves an entirely different set of knowledge and skills (Blanton, 2008). These include looking for patterns, conjecturing and generalising relationships in mathematics. Functional thinking is a process based approach where the focus is shifted from an arithmetic focus to an algebraic focus, and where relationships between the numbers are important (Schoenfeld, 1987; Siemon, Beswick, Brady, Clark, Faragher & Warren, 2015). Blanton *et al*. (2015:512) state that functional thinking entails (a) generalising relationships between co-varying quantities; (b) representing and justifying relationships in multiple ways using natural language, variable notation, tables and graphs; and (c) reasoning fluently with these generalised representations in order to understand and predict the functional behaviour.

The aim of functional thinking is to deepen the conceptual understanding of mathematics and to allow children opportunities to see the generality within mathematics. Day (2017) believes that patterns highlight the structure of numbers, and noticing this structure of arithmetic is the basis of algebraic reasoning. Blanton and Kaput (2011) found that when given ample opportunities to think about functions, children start by using their natural language and through experiences and scaffolding will move towards a more symbolic representation.

Recent research in functional thinking suggests that even young children can engage in this way of thinking and are able to represent, in different ways, how two different quantities correspond involving letters as variables. Pang and Kim (2018) agree and add that learners' algebraic abilities can be enhanced if the appropriate tasks and well-designed curricula are used. Moss and London McNab (2011) found that Grade 2 learners could look at the relation between position numbers and the number of blocks in a growing geometric pattern and develop a function rule. Blanton *et al*. (2015) state that even Grade 1 learners are able to generalise the functional relationship between co-varying quantities, and with further support and well-designed tasks, their level of understanding could improve and become more sophisticated. In a study by Carraher and Schliemann (2016) it was found that children as young as nine years old could use relations to derive other relations. While previous research into young children's algebraic thinking looked at shifting the focus from recursive to functional thinking (Warren & Cooper, 2008), more recent research suggests that younger primary school learners are more able to think algebraically than what was imagined in the past. It is, however, important to note that although young children have the capacity to reason about functions and can develop and use different tools to represent this, this way of thinking does not happen spontaneously and requires well designed activities with constant support and scaffolding from the teacher (Blanton & Kaput, 2011).

Research in the field of algebraic thinking is relatively new and whilst there is no one clear definition of Early Algebra, the work of Kaput (2008) and the subsequent work of other researchers in the field attest to the need for ongoing research of this topic. This study will be focusing on functional thinking, which involves looking for patterns, conjecturing and generalising relationships, as it is a relatively new addition to the younger grades' curriculum and is often poorly taught in primary school (Blanton, 2008).

2.3 Curriculum

A curriculum is often referred to as a formal course, where the specific content or subject matter is emphasised. It is grade specific and includes the recommended pace of lessons as well as the order in which the lessons are to be taught (Powell, 2014). It also includes the experiences presented to the learners to assist in comprehending the subject matter. Lunenburg (2011) suggests that a curriculum can be used to provide the expected learning outcomes of a specific subject. Teachers play a vital role in the shaping of the curriculum but there can be variations in the ways they interpret the curriculum. Remillard and Bryans (2004) believe that in order for the curriculum to be successful, the teacher must engage with and understand it before teaching it to learners. Remillard (1999) suggests that teachers need clear and well-designed curriculum guidance. This type of guidance can help to ensure that teachers do not misinterpret the curriculum or use it incorrectly.

2.3.1 Studies related to curricula and algebraic thinking

Cai and Knuth (2005) suggest that curricula have a direct influence on what children learn and is a contributing factor in the differences between countries' mathematical performance. Furthermore, they offer a useful overview of the development of algebraic thinking in the younger grades from different curricula perspectives: China, Singapore, South Korea, Russia and the USA. Each curriculum was analysed according to the following criteria; goal specification, content coverage and process coverage. Goal specification looked at the aims of developing algebraic thinking within each curriculum and these were compared to the four algebra goals from the National Council of Teachers of Mathematics (NCTM)'s Principles and Standards for School Mathematics in the USA (2000). The NCTM's goals are to understand patterns, relations, and functions; to represent and analyse mathematical situations and structures using algebraic symbols; to use mathematical models to represent and understand quantitative relationships; and to analyse change in various contexts. The content coverage focused on the broader concepts of algebra, including variables; reasoning; patterns and relationships; equivalence of expressions; equation solving; representation and modelling. The third category, process coverage, looked at how the curriculum's design fostered habits of algebraic thinking within the learners. It also focused on how the curricula helped learners use their previous experiences to assist them when doing formal algebra.

The main goal for teaching and learning algebraic concepts appears to remain the same throughout the different curricula. The general aim is to deepen the learners' understanding and to assist them when faced with formal algebra. The Chinese curriculum for younger grades focuses on quantitative relationships. The learners are encouraged to represent these relationships arithmetically and algebraically, later comparing the two representations. The Russian curriculum, based on the work of Davydov, uses concrete work with quantities to develop algebraic understanding. The South Korean curriculum attempts to bridge the gap between arithmetic and algebra by using concrete materials and the Singaporean curriculum uses number pattern activities to encourage learners to make generalisations. Equations are not formally introduced, and instead are introduced with pictures which are used not only to assist the learners in solving mathematical problems, but in developing their algebraic thinking.

There is also research which has been conducted which looks at the treatment of Early Algebra content in the intended curriculum within English and South African contexts (Roberts, 2010). The results from two levels of analysis reveal that a curriculum may include algebra but does not have much substance and alternatively that a curriculum may have a lot of algebra but does not mention it by name. Mc Auliffe (2013) and Vermeulen (2008) agree that the South African curriculum realises the importance of developing algebraic principles which is made evident through one of the content areas: Patterns, Functions and Algebra. Roberts (2010), however, suggests that the way algebra is incorporated in the Foundation Phase curriculum treats it as a stepping stone for formal algebra and not as a tool for developing algebraic thinking. In order to use the mathematics curriculum as intended, the correct classroom atmosphere must also be developed (Vermeulen, 2008). Roberts (2010) believes this can be achieved if teachers are sufficiently guided.

In a study to ascertain how young children develop and express functions, Blanton and Kaput (2004) provided children with word problems where the learners were expected to develop a functional relationship between a random amount of dogs and the corresponding number of eyes and/or tails. They found that in the younger grades the teacher recorded the data as the children counted the visible objects. At this stage, there was no indication that the children were looking for patterns in the data. In the Grade R year equivalent, data were recorded by making dots and stripes. The learners at this level used skip counting to find the total number of eyes and eyes/tails. The children also found and described a basic pattern that the total number will always be even. By first grade, the children recorded their own data. Although they also made use of skip counting, they could also see that the total number of eyes doubles, while the total of eyes and tails triples. The learners at this level were able to identify and describe in natural language an additive pattern. At the second grade level the learners could identify a multiplicative pattern, however, could only express it using words. This showed that these learners were able to read across the data, as opposed to just down the columns. From the third grade upwards the learners could explain and express the multiplicative relationship using words and symbols. As the learners developed in this way of thinking, they required less data to develop an algebraic function. This demonstrates a trajectory of how these young children moved through activities to develop and express functions. Content related to functions in a mathematics curriculum should follow a similar trajectory to develop this skill.

Blanton and Kaput (2004) state that the patterning seen in most mathematics curricula does not aim to develop functional thinking. Although the pattern section requires the learners to find patterns in a set of data, the data usually consists of a single variable. The skills acquired from doing such activities could assist with the development of functional thinking in the higher grades, however, it does not require the learners to predict an outcome. They conclude by suggesting that the pattern section for the younger grades should focus on how two or more quantities relate to each other, instead of simple patterning.

Blanton, Brizuela and Stephens (2016) found that elementary age children were able to participate in sophisticated forms of algebraic thinking, namely generalising, representing, justifying and reasoning with mathematical structure and relationships. They found that carefully designed, continuous interventions could improve a child's early understanding of algebra and could potentially better equip them for the formal algebra later in the school career. Despite the common algebraic difficulties which include the inability to move from recursive thinking to functional thinking, Blanton *et al.*, (2016) learned that young children were more able to think algebraically than what was believed. They conclude by stating that sustained experiences involving the mathematical practices that characterise algebraic thinking, from the beginning of the child's mathematical career, can assist in alleviating the misconceptions of algebra often seen in older learners.

2.3.2 The South African mathematics curriculum

The Curriculum and Policy Statement (CAPS) is the current curriculum in South African schools (SA.DBE, 2011). CAPS outlines the knowledge, skills and values to be taught in schools and incorporates these into a termly and daily teaching plan. The current design is based on social transformation and tries to redress the social and educational inequalities from the past, by providing equal and fair educational opportunities for all. CAPS encourages active learning and aims to have learners involved in the learning process as opposed to rote memorisation of rules. In terms of mathematics, CAPS shows a logical progression of concepts from simple to more complex. It seeks to provide South African learners with a high level of education, comparable to that of other countries. The Department of Basic Education (SA.DBE, 2011) designed the CAPS curriculum with the vision to produce learners who are able to think critically, solve problems and relate this to the challenges of everyday life. It is designed to develop learners who have a deep and conceptual understanding of mathematics and can further their studies in this field (SA.DBE, 2011).

Patterns, functions and algebra is the second content area of the South African curriculum and is split into two sections: geometric patterns and number patterns. In the Foundation Phase (Grade 1-3), the learners complete and extend patterns represented in different forms. By the end of the Foundation Phase, learners are expected to create and describe their own patterns using drawings, shapes or objects. The number pattern section requires learners to copy, extend and describe number sequences within an appropriate number range. These sequences increase in difficulty as the child progresses through the grades. By the end of the Foundation Phase, learners should be able to create and describe their own number patterns. Although patterns were common in younger grades curricula, Blanton and Kaput (2004) argue that in order to encourage algebraic reasoning, these pattern activities must extend further to include opportunities for learners to think algebraically (focus on the relationships, rather than the outcome).

Mwakapenda (2008) highlights the importance of making connections in mathematics. He suggests the CAPS curriculum provides these connections, for example the connection between algebra and functions. He posits this connection is necessary for learners to be proficient in mathematics. In order to work well with shape, space and measurement, learners will need to draw on both their algebraic as well as their geometric knowledge. The ability to make these types of connections is vital if learners are to have a conceptual understanding of mathematical concepts being taught. There is a problem in the teaching and learning of mathematics in South Africa, as learners are ill-prepared and unable to make the connections needed to perform well in mathematics (Mwakapenda, 2008).

The implementation of the South African mathematics curriculum does not specify a prescribed textbook and a variety of different textbooks are available for teachers to select from. Textbooks may vary but they are all required to meet CAPS requirements. The CAPS curriculum includes a clarification section which provides insight into how each concept should be taught and how progression should be addressed (SA.DBE, 2011). This section includes an explanation of models to use and questions to ask. This clarification serves as the support and guidance for the teacher and it is suggested that the specification of the content and the clarification section of the content should be consulted in conjunction with textbooks (SA.DBE, 2011).

2.3.3 The Singaporean mathematics curriculum

Singapore Mathematics is a curriculum with a strong emphasis on problem solving which is an important component of learning mathematics as it requires learners to apply knowledge they have learnt in mathematics to solve a specific problem (Chen, 2008 & Fong, 2004). It covers a small amount of topics in a deep, conceptual manner (Powell, 2014). There are five pillars to Singapore Mathematics: *concepts, skills, process, attitudes and metacognition* represented as a pentagon to show the connections and relationships of the mathematical priorities or pillars of the Singapore Mathematics system (Singapore. Ministry of Education, 2006).

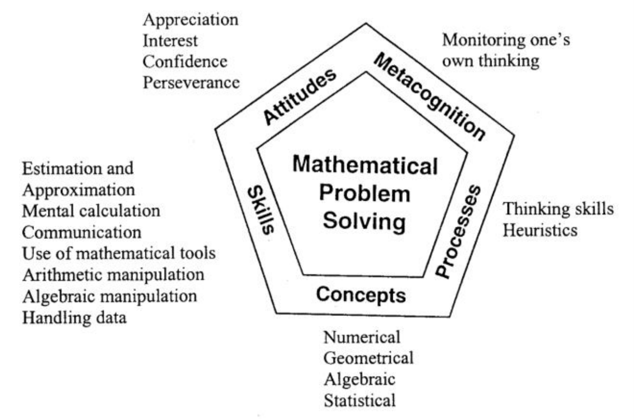


Figure 2.1: The five pillars of Singapore Mathematics (Singapore. Ministry of Education, 2006)

Concepts refer to the mathematical concepts covered across the curriculum. These concepts must be understood conceptually and not just procedurally. A variety of learning experiences are encouraged where concrete apparatus must be used in order for the learner to make meaning of abstract mathematical concepts (Singapore. Ministry of Education, 2006).

Skills are specific to mathematics and are needed to acquire and apply mathematical knowledge. These skills should be taught with an understanding of the underlying principles. In other words, the learner needs to understand why they are doing something as opposed to simply knowing how it is done. When acquiring and applying mathematical knowledge, *process skills* are needed. The processes must be taught explicitly, although younger learners gain these through problem solving. These processes include: mathematical reasoning; communication; making connections; thinking skills; application and modelling (Singapore. Ministry of Education, 2006).

Attitudes refer to the emotional aspects of learning and doing mathematics. These include the learner's confidence in mathematics, their interest and enjoyment of it and their perseverance when solving a problem. A child's attitude is determined by his learning experiences, so these experiences must be fun and meaningful (Singapore. Ministry of Education, 2006).

Metacognition is the ability to control one's thinking ability. This includes knowing which strategy to select when solving specific problems. Learners should be able to solve non-routine problems and should be given the opportunity to discuss their thinking and their solutions (Singapore. Ministry of Education, 2006).

Leinwand and Ginsburg (2007) highlight the close link between the pillars of the Singaporean mathematics curriculum and the five strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition (National Research Council, 2001). Although problem solving is at the centre of the system, the five parts serve as the aims of the curriculum, each one holding equal importance. The pentagon of pillars explains that the process is important, however, if the concepts and skills are also conceptualised.

Furthermore, the Singapore mathematics curriculum highlights the importance of developing an understanding of patterns and algebra in the early grades. Having a conceptual understanding of these aspects of mathematics is vital in learning and understanding other areas of the subject (Singapore. Ministry of Education, 2012). Children are naturally interested in patterns as seen in songs and chants. When they start school, they begin to manipulate and explore patterns in mathematics. In the younger grades the children identify, extend and create their own patterns. Given the variety of learning opportunities provided, older learners can start to analyse more complex mathematical relationships and begin to represent these algebraically. In the Singapore curriculum, patterns and algebra is split into two main sections; patterns and relationships and expressions and equality.

Singapore mathematics has a prescribed textbook for both the learners and the teacher. The learner's book and the teacher's guide have the same activities in the same order. The books are centred on problem solving and model drawing and all new work is based on the knowledge previously taught (Powell, 2014). The teacher's guide is compulsory for the teachers and includes notes for teachers on the concept they are teaching, including questions to ask to elicit specific answers in order to get the learners thinking. Additional resources are included in the guide. These activities develop certain areas within the curriculum and the aim of each activity is explained. The teacher can use these activities to consolidate new knowledge or as revision of work previously covered (Fong, 2004). The Singapore Maths textbooks use minimal text and basic visuals, ensuring the focus is on the mathematics concepts being covered. Throughout the textbooks, links between the concrete, pictorial and abstract are made explicit. The idea of scaffolding is used throughout as the acquisition of new knowledge depends on the conceptualisation of previous knowledge (Powell, 2014). The assumption is that what has already been taught need not be taught again, only built upon (Ginsburg *et al.*, 2005). Hu (2010) highlights the strong emphasis on children understanding concepts moving from concrete, to pictorial to symbolic.

2.4 Textbooks

Garner (1992) identifies textbooks as a critical vehicle used in the acquisition of knowledge in classrooms. He adds that textbooks are so important in some contexts that they can replace the teacher's voice. Haggarty and Pepin (2002) support this by arguing that learners spend so much of their time working with textbooks, they can be seen as one of the main translations of a curriculum. Textbooks are of utmost importance as they dominate what is learnt in the classroom and help teachers structure their lessons (Apple, 1992). Most textbooks used in classrooms do not, however, fulfil their purpose, show a lack of coherence and do not make explicit connections between the main ideas (Velverde & Schmidt, 1998).

Given the importance of textbooks in relation to teaching and learning, Sood and Jitendra (2007) studied information in textbooks to determine whether they were adequate to serve as the main conveyer of information. They found that traditional textbooks were criticised for being repetitive, unfocused and undemanding. They suggest that if a mathematical textbook is to be used optimally to enhance learning, four important features have to be present. Firstly, there needs to be a clear and meaningful development of mathematics in the textbook as well as materials and activities that promote real world experiences. This will encourage a conceptual understanding as well as the understanding that mathematics exists in the everyday context. Secondly, the tasks in the textbook should include multiple models and the connections between these models should be explicitly stated. Thirdly, mathematics textbooks should be used to provide opportunity for learners to analyse their own performance in a stress free environment, where they feel safe to take risks and make mistakes. Finally, it was found that in order to enhance learning, the textbook should provide ample opportunity for the learners to apply their newly learned skills by means of scaffolding. Overall, many textbooks provide little support for the teachers and make the assumption that all teachers have deep mathematical knowledge and are confident in teaching the content. Textbooks also need to provide support for teachers in understanding the content and help to create an environment conducive to learning.

Schmidt, McKnight, Valverde, Houang and Wiley (1997) conducted research into the analysis of mathematical text and suggest that content should be classified according to specific criteria. The criteria consist of the *topic complexity*, the *developmental complexity* and the *cognitive complexity* of the mathematical text. The *topic complexity* refers to the topics that are emphasised, when in the process this is done and what the conceptual demands are. The *developmental complexity* refers to the way in which the topics are sequenced and developed across lessons and throughout the whole curriculum, including when topics are revisited. The *cognitive complexity* is the aim of the lesson or what the learners learn from the specific topic.

Vermeulen (2016) conducted an analysis of three South African Grade 4 mathematics textbooks to ascertain how well they develop algebraic thinking. He found that the authors seemed to understand the expectations of the curriculum in terms of algebraic thinking and two textbooks offered some opportunity for this development to occur. The third textbook did not promote the development of algebraic thinking and the sequence of the activities in all three textbooks was a concern.

Twohill (2016) highlights two ways of thinking when posed with constructing a general rule for a pattern problem. An explicit approach (or functional thinking) is when a relationship between a term and its position in the sequence is established. A recursive approach (or recursive thinking) is when a comparison is made between consecutive terms in the pattern in order to find the relationship between them. This relationship is then used to find the next term in the sequence. When children are posed with a pattern problem, their natural instinct is to use the recursive approach which does not assist in developing the child's conceptual understanding of patterns and the relationships that exist between the terms and their positions. The child may need to be encouraged and guided toward thinking explicitly when finding the relationships within patterns. The two approaches are not considered hierarchical but rather complimentary and specific. Well planned materials are required in order to elicit the use of each approach (Watson, Jones & Pratt, 2013). Twohill (2016) explains that textbooks and activities should not only favour the recursive approach to patterns. Learners need exposure to both approaches and their thinking should be scaffolded and developed to the point where they are able to choose the most appropriate approach depending on the situation.

2.5 Theoretical framework

This section provides an overview of the Early Algebra framework developed by Kaput (2008) which has been used to create a content focus for this study and two current frameworks from empirical research used to characterise different levels of children's functional thinking (Twohill, 2013; Blanton *et al.* 2015).

Kaput (2008:10-14) was one of the first to develop a framework to think about Early Algebra. It comprises of two core aspects related to the reasoning processes and three algebraic strands of Early Algebra. The first core aspect involves using symbols to generalise and the second core involves acting on symbols and following rules. The three algebraic strands, two of which were discussed earlier (2.2.3), are defined as follows:

1. Generalised arithmetic – generalising number properties and relationships, moving towards the notion of a function; including commutativity, inverse operations and the equivalence of the equal sign. Initial symbolisation that does not include numerals are used to compare quantities.

2. Functional thinking – seeing algebra as the study of functions and relations which includes noticing change and representing these using tables, graphs or function machines.

3. Modeling – using algebraic reasoning and language to understand and interpret a situation or problem.

This study focuses on the second aspect of Early Algebra: functional thinking, which involves children using symbols to generalise.

This following section presents two different frameworks developed by Twohill (2013) and Blanton *et al*. (2015). The framework by Twohill provides a starting point for understanding how children move between different levels of algebraic thinking when solving pattern problems. The Blanton *et al*. (2015) model provides more detail on each of the levels. Twohill's (2013) framework consists of five growth points to investigate and explore the level of algebraic thinking skills of children in the primary school. Her framework is not dependent on the child's age or grade and aims to track the developmental trajectory of a primary school child.

Table 2.1: A framework of growth points in algebraic reasoning (Twohill, 2013)

Growth point	**Explanation**
Growth point 0: Pre-formal pattern	At this growth point, the child does not possess a formal understanding of pattern and he is unable to identify a repeating term in a pattern.
Growth point 1: Informal pattern	At this level, the child understands pattern and is able to identify a commonality within the pattern. The child can copy, extend and complete different types of patterns (visual spatial, numeric, repeating and growing).
Growth point 2: Formal pattern	At this stage the child is able to describe a pattern in words and can reason to offer a possible near term.
Growth point 3: Generalisation	The child is now able to correctly identify a near term and can explicitly describe the pattern. The child can offer a possible far term with reasoning.
Growth point 4: Abstract generalisation	At this growth point, the child is able to describe the pattern and generate and explain the rule. This rule can be described as an expression using symbols and can be used in order to correctly identify a far term.

Twohill (2013) explains that the first three growth points use the skills involved with pattern solving and that patterns and patterning are the vehicles for introducing algebra in the early years of school. Blanton *et al*. (2015) used levels of sophistication to analyse children's thinking when asked to generate and organise data for a given scenario and then to develop a function rule. They focused on the type of relationships children notice when solving problems and did not focus on the actual representations used.

The Blanton *et al* (2015) model has 8 levels of sophistication. Children do not need to go through each one in a linear fashion in order to proceed to the next level. Instead, some children skip levels while others move through multiple stages during one specific problem.

Table 2.2: Levels of sophistication in children's thinking about generalising (Blanton *et al*., 2015)

Level number and name	Characteristics of level	Example
Level 1 Prestructural	Did not describe or use any mathematical relationship. Could not see recursive pattern. Learner might describe a non-mathematical pattern they have observed (not focused on pattern or relationship)	Guesses random answer- cannot explain why.
Level 2 Recursive-particular	Learner conceptualises a recursive pattern (vertically) but does not see it as a generalisation. Learner's understanding of the relationship is still represented as a counting process (dependent on the previous value)	Counting The learner says it's 2,4,6,8 and NOT that 2 is added every time
Level 3 Recursive-general	Also seen as a recursive pattern but now has a general rule for that pattern. Similar to level 2 but learner can tell you that the next value will be 2 more than the last. Still looking at the vertical.	The learner says you count 2,4,6,8…you add 2 each time.
Level 4 Functional-particular	Learner conceptualises a functional relationship as a set of particular relationships using specific values that correspond. Learner can explain the relationship specific to this case but not as a general rule. Can apply the rule and give output only if input is supplied.	The learner can tell you 2 dogs have 4 eyes because 2 doubled is 4. Can't tell you how many eyes for x amount of dogs.
Level 5 Primitive functional-general	Learner conceptualises a functional rule across a set of cases but representations are primitive. Learner can describe a general relationship but cannot articulate the relationship that identifies the specific quantities or the mathematics behind it.	The learner can complete the data in a table but cannot verbalise the rule

Table 2.2: Levels of sophistication in children's thinking about generalising (Cont.)

Level 6 Emergent functional-general	Learner's generalised functional relationship contains some key attributes but is still incomplete. Either the variables are left out or leaves out the "mathematics". Learner sees the generalised relationship between the quantities but it is primitive and is described as an expression.	The learner can say value "increases by 1cm" but doesn't name the two variables (e.g The person's height increases by 1 to get his height with a hat on)
Level 7 Condensed functional-general	Learner generalises the rule between two arbitrary quantities using both words and variable notation. The rule is expressed as an equation and learner uses sentences that identify both quantities and their relationship. Learner's thinking is more sophisticated than previous level.	The learner can tell you the rule in words, using the quantities and the function in a general form.
Level 8 Function as object	Learner generalises the rule as in previous level and now perceives boundaries concerning generality of the relationship. The learner knows that the rule does not hold if the scenario changes and no longer sees the rule as a process to get the output, but as a rule that can be transformed if necessary (to suit the new scenario)	Using the learner's rule, the teacher could change the scenario slightly and the learner would be able to adapt the rule to suit the problem.

Based on Kaput's (2008) formulation of Early Algebra, Blanton, Brizuela and Stephens (2016) developed four critical mathematical practices that characterise early algebraic thinking. These are important as they help to outline what mathematical practices are useful in the development of levels of algebraic thinking and will also inform the analysis of the learner responses. These are *generalising, representing, justifying and reasoning* with mathematical structure and relationships.

Generalising mathematical structure and relationships is of utmost importance as it forces the learner to focus on the relationships that hold specific mathematical structures together as opposed to focusing on the specifics of particular arithmetic instances. A learner's ability to generalise highlights his understanding of the situation, his ability to connect it to his prior knowledge and his ability to draw conclusions by making a generalised claim. Representing the mathematical structure or relationships is seen by Kaput (2008) as being of equal importance to generalising. The use of representations, whether conventional or non-conventional, moulds the learner's understanding of the problem and is proof of the generalisation he has made. It also shows learners that a specific action can apply to an infinite amount of cases and not just one particular scenario. The act of justifying generalisations involves learners developing mathematical arguments to defend or disprove the validity of a generalisation. Blanton *et al*. (2016) state that justifying runs alongside generalising as the learners first explore the situation, then express their generalisation and finally attempt to justify their own rule. Through the process of justifying, the generalisation itself is often strengthened. The types of justifications learners make will vary, depending on their understanding of the scenario, the generalisation itself and their mathematical knowledge and confidence. After the generalisation, representation and justification processes, algebraic thinking involves reasoning with the generalisations as objects. Learners use their generalisations as true objects and apply it when attempting to solve new problems. This can be seen as a cognitive process where the generalisation has been re-evaluated in the child's thinking.

The figure below (2.2) shows how the frameworks developed by Twohill (2013) and Blanton *et al*. (2015) link and overlap. Both frameworks are explained and reviewed. The Twohill framework does not provide enough detail in terms of analysis for this study, and therefore the levels of sophistication by Blanton *et al*. (2015) was adopted as the theoretical framework underpinning this research. This will be used to analyse the textbook activities and responses from teachers and learners from the data collection.

Growth point	Explanation
Growth point 0: Pre-formal pattern	At this growth point, the child does not possess a formal understanding of pattern and he is unable to identify a repeating term in a pattern.
Growth point 1: Informal pattern	At this level, the child understands pattern and is able to identify a commonality within the pattern. The child can copy, extend and complete different types of patterns (visual spatial, numeric, repeating and growing).
Growth point 2: Formal pattern	At this stage the child is able to describe a pattern in words and can reason to offer a possible near term.
Growth point 3: Generalisation	The child is now able to correctly identify a near term and can explicitly describe the pattern. The child can offer a possible far term with reasoning.
Growth point 4: Abstract generalisation	At this growth point, the child is able to describe the pattern and generate and explain the rule. This rule can be described as an expression using symbols and can be used in order to correctly identify a far term.

Level number and name	Characteristics of level	Example
Level 1 Prestructural	Did not describe or use any mathematical relationship. Could not see recursive pattern. Cd might describe a non-mathematical pattern they have observed (not focused on pattern or relationship)	Guesses random answer- cannot explain why. Diagonal numbers in a table are the same (no pattern)
Level 2 Recursive-particular	Cd conceptualised a recursive pattern (vertical), but do not see it as a generalisation. Cd's understanding of the relationship is still represented as a counting process (dependent on the previous value)	Counting. Cd says it's 2,4,6,8 and NOT that you add 2 every time
Level 3 Recursive-general	Also seen as a recursive pattern but had a general rule for that pattern. Similar to level 2 but now cd can tell you that the next value will be 2 more than the last. Still looking at the vertical.	Cd says you count 2,4,6,8...you add 2 each time.
Level 4 Functional-particular	Cd conceptualised a functional relationship as a set of particular relationships using specific values that correspond. Cd can explain the relationship specific to this case but not as a general rule. Can apply the rule and give output ONLY if input is supplied.	Cd can tell you 2 dogs have 4 eyes because 2 doubled is 4. Cant tell you how many eyes for x amount of dogs.
Level 5 Primitive functional-general	Cd can conceptualise a functional rule across a set of cases but representations are primitive. Cd can describe a general relationship but can't articulate the relationship that identifies the specific quantities or the mathematics behind it.	Cd can complete the data in the table etc but cannot verbalise the rule
Level 6 Emergent functional-general	Cd's generalised functional relationship contains some key attributes but is still incomplete. Either leaves out the variables or leaves out the "mathematics". Cd sees the generalised relationship between the quantities but it is primitive, describes it as an expression.	Cd can say it "increases by 1" but doesn't add in the two variables (e.g The person's height increases by 1 to get his height with a hat on)
Level 7 Condensed functional-general	Cd can generalise rule between 2 arbitrary quantities using both words and variable notation. They expressed the rule as an equation and used sentences that identified both quantities and their relationship. Thinking is more sophisticated than previous level. Were flexible in their thinking	Cd can tell you the rule in words, using the quantities and the function in a general form.
Level 8 Function as object	Cd can generalise rule as in previous level and now perceives boundaries concerning generality of the relationship. Cd knows that the rule wouldn't hold if the scenario changed. No longer see the rule as a process to get the output but as a rule that can be transformed if necessary (to suit the new scenario)	Using the child's rule, teacher can change the scenario slightly and the cd can adapt the rule to suit.

Figure 2.2: The development of the frameworks

(Twohill 2013; Blanton et al. 2015)

for Early Algebra but this study focuses specifically on patterns and functional thinking in alignment with curriculum requirements for the younger grades. Research related to Early Algebra, curriculum and textbooks highlight the importance of guidelines for teachers and the impact on mathematical performance. There are different frameworks available for the analysis of children's algebraic thinking and the Blanton model provides a detailed characterisation of each level. This will be used further to form the theoretical underpinning of this research and to inform the analysis of data.

CHAPTER THREE

METHODOLOGY

3.1 Introduction

This chapter describes the design and methodology of the study. It is a qualitative study using a case study approach to address the main research question and sub-questions. The site and sample consist of learners from two Grade 3 classes, at two primary schools in Cape Town, using either the South African CAPS guidelines or the Singapore mathematics curriculum. Data were collected using a variety of methods which include document analysis, written tests, semi-structured interviews and focus group interviews. The purpose of each data collection method and the process involved in the collection and analysis of this data are explained further in this chapter.

Creswell (2014:12) defines a research design as “the type of inquiry within qualitative, quantitative, and mixed method approaches that provide specific direction for procedures in a research design”. The design can involve a quantitative, qualitative or mixed methods approach. Quantitative research is defined as examining relationships between variables in order for data containing numbers to be analysed (Creswell, 2014). A qualitative study is defined by Maxwell (2013) as an inductive, open-ended approach with a goal of providing a better understanding of a phenomenon and other perceptions of it. Qualitative data is usually not numeric and is comprised of descriptive information (Yin, 2014). Creswell (2014) defines qualitative research as exploring reasons behind a social or human problem. The mixed methods approach combines the structure of quantitative research with the flexibility of qualitative designs to achieve a broader understanding of the study (Creswell, 2014).

This research makes use of a qualitative design as the researcher investigated how two specific mathematics curricula provide for the development of algebraic thinking. Curriculum documents and textbooks, along with semi-structured interviews with the class teachers were analysed to understand the content and instruction of each curriculum. Learner test results and focus group interviews were then observed and analysed to understand how the learners perform when solving functional thinking type problems. In this qualitative study, I aimed to ascertain the opportunities offered within each curriculum for the development of algebraic thinking.

3.2 Design and methodology

There are different methodologies that can be used in qualitative research studies. These include action research, comparative research, evaluation, experiments and case studies (Thomas, 2011). This research study is qualitative in nature and makes use of the case study methodology. Yin (2014:16) defines a case study as follows:

> A case study is an empirical enquiry that investigates a contemporary phenomenon (the "case") in depth and within its real-world context, especially when the boundaries between phenomenon and context may not be clearly evident.

A case study is an enquiry into a research topic within its real life context (Nieuwenhuis, 2012). It aims to provide a deeper, more descriptive understanding of how participants relate to the topic and provides access to their lived experiences (Bertram & Christiansen, 2014). Rule and John (2011) define a case study as a structured and deep investigation of a specific event within its context, with the aim of gaining knowledge about it. A case study allows the researcher to focus on a particular section, making the study more manageable (Rule & John, 2011). The limitations of a case study include the argument that case studies are limited as it is only one study and the findings cannot be generalised to explain a pattern about the case. Rule and John (2011) argue that this is not the intention when choosing a case study in one's research. They further explain the choice of methodology must fit the purpose of the study.

The aim of this study is to understand how two different mathematics curricula provide for the development of algebraic thinking. This includes the role of the curriculum documents, textbooks and teachers in this development. A case study was selected as it allows for a deeper understanding of a specific event, within context. Although a case study can be seen as limited as it cannot be used to generalise about the broader community, this was not the intention for this research study. This study aims to develop a deep understanding of how these curricula, textbooks and teachers, in these specific classes, assist in and provide for the development of algebraic thinking. For this reason, a case study approach suited the purpose of the study.

3.3 Research plan

The data used in this study was collected using different collection methods. The curriculum documents and textbooks were sourced and collated in June 2016. The plan was to collect the remaining data within the third term of the school year (July – September). Due to time constraints, the test data were collected during November 2016. A pencil and paper test was designed for all the learners to complete. This test was administered by the class teachers in early November 2016. The learners were given 60 minutes to complete the test which was done under normal test conditions in the classroom. Following this, the teachers were interviewed and the learners participated in focus group interviews. Both of these interviews were conducted later in November 2016. The teacher interviews lasted approximately 30 minutes each and, for time purposes, were done on different days. The focus group interviews consisted of eight learners per class. These learners were purposively selected by the class teachers to ensure the focus groups represented a group of mixed mathematical ability. The group interviews in each school were conducted on different days and each group took approximately 45 minutes, to allow time for further discussions, if necessary.

3.3.1 Pencil and paper test design

A pencil and paper test (see Appendix 1) was designed comprising of a variety of algebraic thinking questions. Although the focus of the study is functional thinking, both generalised arithmetic and functional thinking type questions were included in the test in order to understand the learners' overall performance in Early Algebra activities, highlighting the areas of weakness. The test was designed using examples from ANA papers and grade specific textbooks. The same test was completed by both classes under the supervision of the class teachers and it contained nine question items.

The first three test items were counting patterns in which the learners were given a series of patterns with missing numbers. They had to fill in the missing numbers to correctly complete the sequence. The learners were then asked to explain what they noticed about the patterns. Item one's sequence was counting backwards in multiples of 5. Item two required the learners to count forwards in multiples of 4. The third item required the learners to count in multiples of 4 and then 40 to complete both sequences. The learners were then asked what they noticed about the patterns, to ascertain whether the learners were able to see a relationship between counting in 4's and counting in 40's. Blanton and Kaput (2004) highlight the importance of patterning and state that the patterns given in younger grades should focus on how quantities relate to each other. Only when this is achieved, as opposed to simple patterning, does it become a tool which will assist in the development of functional thinking required in later grades.

Items four and five focused on the notion of equivalence. In item 4, the learners were given a series of equations and they had to state whether or not they were true or false. There was also an “I don’t know” option to eliminate guessing. Item 5 was slightly different as the learners were given a series of equations with a number or an operation symbol left out in each equation. The learners had to fill in the missing symbol in order to make the statement true. Equivalence was included in this test because it is seen as one of the foundations of Early Algebra. Matthews, Rittle-Johnson, McEldoon and Taylor (2012) suggest a direct link between the notion of equivalence and algebraic thinking.

Working with a variable formed item six in which the learners had to multiply a number by 2, add 4 and were told the answer would be 30. They had to determine the number and then explain how they did this. They were asked to explain their thinking, allowing the researcher to ascertain whether they were using algebraic thinking to solve the problem or if they used a guessing method. Working with variables is seen as one of the big ideas of Early Algebra as it requires the learners to indirectly apply algebraic rules (Greenes, 2004). In order to solve for the variable, the learners have to engage in various processes in which they can make use of rules such as the commutative property and equivalence.

Item seven involved a function table using the function y= 3x in which the learners had to complete the table and then answer questions predicting a further term. In order to do this, they would have to demonstrate an understanding of the relationship between the dependent and the independent variables. This was included as it serves as a way to test the learners’ functional thinking as they were required to look at the relationship between the numbers in order to answer the questions (Blanton *et al.*, 2015). It relies on functional thinking as opposed to a recursion. A functional approach is one where one looks across the data and the focus is on the relationship between the dependent and independent variables. A recursive approach focuses on relationships within one set of data (the independent variables) and focuses on finding the next term by using the previous one (Twohill, 2016). An example would be adding 3 each time.

Item eight was a geometric pattern in which squares were built using sticks and the function y= 3x +1. The learners were shown the first three pictures in the sequence and were then asked a series of questions. The first question required the learners to draw the next picture in the sequence. The next question required the learners to draw picture 6 of the sequence, providing them the opportunity to analyse the function and how the pattern was growing. The learners were then asked to predict how many sticks would be required to build a far term, i.e. picture 20 and were asked to explain how this was achieved. The aim of this was to ascertain the methods the learners were using to answer prediction questions. The researcher wished to ascertain whether learners were able to use information from a special case and apply it to a general case. Geometric patterns are useful when wanting learners to describe patterns and make generalisations about them (Charles & Carmel, 2005). By working with geometric patterns, learners can begin to make generalisations by describing the patterns they notice, thus developing algebraic thinking.

Test item nine was a function table, with $y = 3x+1$ as the function rule. This question used the same function rule as question eight but used a different representation. The aim was to understand how the learners coped with the change in representation as opposed to the complexity of the function rule. This question required the learners to find the missing input and output values. This item was selected to determine the learners' ability to apply a rule to a set of values.

3.3.2 Teacher interviews

Two teachers, one from each Grade 3 class, were interviewed in an attempt to ascertain what they understand about patterns and pattern activities. The aim of including the teachers in the research was to understand how they interpret and use their curricula to provide opportunities to develop algebraic thinking. The teachers were asked questions about the types of pattern activities they do with their learners and how these lessons are taught. Furthermore, they were questioned on how they use their curriculum documents and textbooks to plan for pattern lesson and whether they thought patterns were an important component in the mathematics curriculum and for what reasons.

3.3.3 Focus group interviews

Eight learners, of mixed abilities, per class were chosen to form part of a focus group task-based interview. The aim was to probe more deeply into the learners' understanding of patterns and how they solved these types of problems. The focus groups interviews were audio recorded while they worked through the functional thinking tasks. The problems posed to the learners were pattern activities which included prediction questions. One question involved finding a relationship between the number of dogs and the number of eyes ($y = 2x$), while the other was a geometric pattern involving matches to build squares ($y = 2x + 1$). These questions were selected as the focus is on functional relationships and the learners finding and describing the relationship in each question using different types of pattern tasks.

3.4 Research methods (data collection methods)

3.4.1 Site

This research took place in two primary schools in Cape Town, South Africa. Both schools are private schools and are classified as quintile 5 schools under the Western Cape Education Department. Both schools are situated in higher income areas and are fully resourced. One school follows the Singapore mathematics curriculum while the other follows the South African CAPS curriculum. Both schools have used the same curriculum for at least three years, to ensure that the current Grade 3 learners have been exposed to the same system since Grade 1. One school consists of boys and girls, while the other consists of only girls. Both primary schools have classes from Grade R to Grade 7. The Language of Learning and Teaching (LoLT) of both schools is English. The schools were chosen because they are similar in resources, ensuring a fair comparison.

3.4.2 Sample

Thomas (2011:62) defines a sample as a representation of a wider population. The population is the entire group of people in whom the researcher is interested in studying. Three different types of sampling can be used in research, namely probability sampling, convenience sampling and purposeful sampling (Maxwell, 2009:235). Maxwell (2009:235) explains that qualitative research normally makes use of purposeful sampling and describes it as follows:

> This is a strategy in which particular settings, persons, or events are deliberately selected for the important information they can provide that cannot be gotten as well from other choices.

This study makes use of purposeful sampling as the schools, teachers and learners were chosen deliberately. These schools share similar contexts and each use a different curriculum. The sample for this study consisted of all the learners and their teachers from one Grade 3 class at two different schools. One class had 28 learners while the other had 24 learners. Two classes were chosen to make the study more manageable. The Grade 3 classes were selected as they are at the end of the Foundation Phase and would have been exposed to their respective mathematics curriculum for the duration of their formal school careers. This would provide the researcher with an idea of how the classes perform.

3.4.3 Data collection methods

The data collection methods used to answer the research question included document analysis, a written test for the learners, teacher interviews and two focus group interviews which were audio recorded. In the section that follows, each data collection method will be explained and reasons for why they were chosen are be given.

Main research question:

How do the South African and Singapore Mathematics curricula provide for the development of algebraic thinking in the Foundation Phase?

Table 3.1: Research study sub-questions

Research sub-question	Data collection methods
1. How do the CAPS and SM curricula and textbooks provide for the development of algebraic thinking in the Foundation Phase?	Document analysis on curricula and textbooks
2. What are the teachers' perceptions of the development of algebraic thinking within each curriculum?	Teacher semi-structured interviews
3. How do learners using CAPS and SM curricula solve pattern problems?	Written pencil and paper test Focus group interviews

3.4.3.1 Document analysis

Olsen (2012) explains that document analysis involves using documents pertaining to the study which can be accessed from the workplace, library or the internet. Each curriculum as well as textbooks from both Singapore Maths and CAPS were reviewed and compared to the Blanton levels outlined in the theoretical framework. These were used to inform the design of the test and focus group interview questions. The curriculum documents and textbooks, along with the Annual National Assessment (ANA) question papers were perused and examples from these were used and changed in the creation of the pencil and paper test.

The curriculum documents and textbooks were used to investigate what level of Early Algebra was expected from each curriculum. The content and objectives from each curriculum were analysed to better understand what was required for teaching. The textbooks followed by each class were used to inform the researcher of the types of activities the learners were exposed to as well as the learning trajectories that were being followed.

Using tests in research provides the researcher with a powerful method of data collection (Cohen *et al*., 2007). Johnson and Turner (2003) explain that although tests are often used in quantitative research, they can also be used in qualitative data where experimental researchers often design their own tests. For the purpose of this study a quantitative test was used for qualitative purposes.

As mentioned in 3.3.1, a test was given to each class to ascertain their overall algebraic knowledge and skills. It included both generalised arithmetic and functional thinking type questions and was completed individually by the learners under the guidance of their teachers. The researcher designed the test and included some open-ended test items to probe the children's algebraic thinking (Johnson & Turner, 2003). More details regarding the design of the pencil and paper test can be found in section 3.3.1.

The teachers were instructed not to assist the learners. If, however, a learner became anxious, the teachers were allowed to explain the question to that specific learner but had to indicate this by writing on the question paper. This approach was used to reduce learner anxiety which could impact on the overall performance.

3.4.3.2 Interviews

Yin (2014) describes interviews as one of the most important sources of information in a case study. He explains that although the aim of the interview is to gather information to answer a specific question, the interview should resemble a conversation. The questions should flow and not be inflexible. Bertram and Christiansen (2014) refer to this as semi-structured interviews. Johnson and Turner (2003) add that interviews allow the researcher to probe for clarity or more information when necessary. A disadvantage of interviews is that they are time consuming to set up and carry out (Hamilton & Corbett-Whittier, 2013).

The Grade 3 class teachers were interviewed as they act as the link between the curriculum and the child's understanding. The interviews were recorded, with permission, in order for transcription to take place at a later stage. Porter (2006) states that a curriculum can be classified into four categories: intended; enacted; assessed and learned curriculum. The intended curriculum outlines the knowledge and skills which the curriculum aims to teach. The enacted curriculum involves the knowledge and skills that are actually taught in the classroom and the methods used by the teacher to deliver this. The assessed curriculum is the parts of the curriculum that are taught and the learned curriculum is the information and skills acquired by the learners. The questions posed to the teachers aimed to explore their enacted curriculum within their respective classrooms to gain a deeper understanding of how they understand pattern activities and how this impacts their learners' performance.

Table 3.2 Teacher interview questions

Interview questions for teachers
1. What types of pattern activities do you do with your class and how are these taught?
2. How do you use the curriculum documents and textbooks in your teaching of patterns?
3. Do you see the pattern section as an important component of the mathematics curriculum? Why do you think it is important and how (if at all) do you feel it is connected to other areas of mathematics?

The questions posed to the teachers related to pattern work and to the types of pattern activities being done in class. The topic of patterns was chosen as pattern activities have the potential to develop functional thinking and are found in both curricula. The teachers were asked how they taught these pattern lessons and whether or not they thought patterns were an important section of the curriculum. Kieran *et al.* (2016) suggest patterns is one of the aspects at the heart of the development of algebraic thinking. Working with pattern allows and encourages learners to begin looking at relationships in order to later describe and generalise them (Charles & Carmel, 2005).

3.4.3.3 Focus group interviews

Johnson and Turner (2003) suggest that focus group interviews can be beneficial to research as it allows the researcher to gain comprehensive information about how the participants think about an issue. Focus group task-based interviews allowed the researcher to probe for further information related to learners' functional thinking. A disadvantage of focus group interviews is that it could potentially be dominated by one or two participants. Yin (2014:112) warns about "*reflexivity"* when conducting a focus group interview. This occurs due to the conversational nature of the interview. Both the researcher and the participants can influence each other with the questions and responses they provide. The duration of the focus group interview is also important and could influence the level of reflexivity if the interview is too long or too short. Yin (2014) suggests that the researcher should be aware of this possibility and be sensitive towards it.

The patterns tasks used in the focus group interviews were completed one at a time to give opportunity for group work and discussions. The first pattern task was within a context familiar to the learners i.e. dogs and eyes, helping them to visualise the story ($y = 2x$), while the second task provided the learners the opportunity to physically build the first few terms in order to gain a deeper understanding of the structure of the pattern problem ($y = 2x + 1$) (Blanton *et al*., 2015). The pupils were given different types of functions to probe their understanding of different pattern relationships.

The learners were encouraged to work in pairs and to verbalise their strategies. Morgan (2007) states that focus groups which are less structured can be a useful tool in research. This is because the participants can speak freely without too many boundaries. I joined discussions and asked probing questions to keep the learners focused and to understand their algebraic thinking. The focus group interviews were audio recorded to assist later in the analysis phase. Video recordings were chosen, as audio recordings do not record silent activities, such as the procedure being written by the learner (Hopkins, 2014). Taking notes alone would result in the researcher possibly missing important moments and discussions. The responses from the focus group interviews were transcribed, coded and analysed. This process will be further explained in the data analysis section of this study.

3.5 Data analysis

Data analysis is the process in which the researcher makes sense of the data that has been collected. This section will provide insight into how each set of data were analysed and will explain the analytical tools used. Cohen *et al*. (2007:461) describe data analysis as

> "…organising, accounting for and explaining the data in terms of the participants' definitions of the situation, noting patterns, themes, categories and regularities."

Two frameworks were used as the analytical tools for the data analysis process. The first is a framework developed by Tarr *et al*. (2006) for the reviewing and selecting of mathematical textbooks and the second framework the levels of sophistication by Blanton *et al*. (2015). The curriculum documents and textbooks were analysed using both frameworks while the written tests, teacher interviews and focus group interviews were analysed using only the levels of sophistication of algebraic thinking (Blanton *et al*., 2015).

Yin (2014) suggests five techniques for data analysis. These are pattern matching, explanation building, time series analysis, logic models and cross-case synthesis. Pattern matching is useful in a case study as it matches trends found in the data to those designed by the researcher at the start of the study. If these patterns match, it increases the internal validity of the study. Explanation building is a more in-depth form of pattern matching and involves building an explanation about the case. Time series analysis and logic models are techniques more commonly used in experiments and cause-effect studies respectively. These techniques were not suited to this research study and were therefore not used. Cross-case synthesis applies only when more than one case is being analysed and was not used in this study. Pattern matching and explanation building were used in this study by finding and explaining patterns within the data and building an explanation about the case from this. Although these explanations are often in narrative form, Yin (2014) states that good explanations are ones which show links to theory (Yin, 2014).

3.5.1 Teacher interviews

The teachers' responses were transcribed and summarised to establish an overall view on what each teacher said and thought about patterns. The responses from the teachers were grouped together in terms of what was similar and what was different. These responses were then categorised according to the levels of sophistication (Blanton *et al*., 2015). For example, when asked about pattern activities, one teacher explained that the learners would write a number pattern on a board and then erase a few terms. Another learner would then have to complete the sequence by filling in the missing terms. This response was classified as *recursive general* as it focuses on the idea of a recursive pattern in a generalised form, where the learners use a rule to find specific values, e.g. add 5 each time.

3.5.2 Document analysis

The framework used for the analysis of the curricula documents and textbooks is comprised of three aspects, namely: *mathematical content analysis, instructional focus and teacher support* (Tarr *et al*., 2006).

Mathematics content analysis (MCA) defines how well a textbook supports the expectations of the curriculum: it involves looking at a combination of the development of skills, conceptual understanding and mathematical processes. MCA tracks the progression of topics across grades and how engaging the activities are for the learners.

Instructional focus (IF) examines how a textbook's activities develop mathematics as a 'way of thinking'. IF highlights classroom discussions and debates and looks into whether these are being used as a teaching and learning tool. Lastly, IF looks at how a textbook's activities develop a learner's enquiry, critical thinking, problem solving and sense-making.

Teacher support (TS) focuses on whether the materials provide teachers with the opportunity to increase their own knowledge and understanding of the content. This includes informing the teachers about the learners' expected prior knowledge, as well as common misconceptions to be aware of.

The curriculum documents and textbooks from each curriculum were used to provide examples of the types of questions and activities the learners are exposed to. The curriculum documents and textbooks were analysed using the frameworks by Tarr *et al*. (2006) and Blanton *et al*. (2015). Firstly, the documents were analysed according to each key idea of the framework by Tarr *et al*. (2006) to ascertain the content, instructional focus and level of support offered by each curricula's documents and textbooks. Secondly, the mathematical content related to functional thinking found in each curriculum and textbook was analysed further by using the framework by Blanton *et al*. (2015). The functional thinking activities were classified according to the levels of sophistication (Blanton *et al*., 2015). This second level of analysis was necessary as it provided insight into the levels of the activities the learners in each group are exposed to.

The pencil and paper tests were analysed using the framework by Blanton *et al*. (2015). Firstly, the tests were marked and graded according to correct and incorrect answers. These were tabulated and analysed to highlight the strengths and weaknesses of each group. This first level of analysis showed which questions needed to be analysed further in order to understand how the learners were thinking when answering the questions. Secondly, a selection of questions were analysed further based firstly on low levels of performance and secondly on a large discrepancy in performance between the two groups. The responses were assigned a level according to the methods used by the learners.

3.5.3 Focus group interviews

The focus group interviews were reviewed and transcriptions reread. The methods and responses from the interviews were analysed using the Blanton levels of sophistication. Below is an example of how a response was classified. The conversation below relates to the dogs and eyes problem during a focus group interview:

Interviewer: Okay, so if I had to make up a rule to help me always find out the number of eyes, what would it be?

Learner 1: It's counting in twos

Learner one's response was classified as recursive particular as it demonstrates the learner's understanding of pattern as a counting sequence and not as a relationship.

The data analysis process allowed for patterns within the different sets of data to emerge. These patterns helped clarify how these different curricula provide for the development of algebraic thinking. The documents, textbooks and teachers' responses highlighted the input the learners receive, while the written tests and focus group interviews demonstrated how the learners perform when solving functional thinking pattern problems. The findings from this analysis process are explained and discussed in the next chapter.

3.6 Validity of the study

In order to ensure the quality of case study research, certain measures need to be taken. Rule and John (2011) refer to these as validity, reliability and generalisability. Guba (1981) moved away from the traditional ideas of validity and reliability and offered the idea of trustworthiness of data (Rule & John, 2011). In order to achieve trustworthiness in the data, the *transferability, credibility, dependability and confirmability* must be the focus. *Transferability* involves the validity of the study and is seen as an alternative for generalisability. *Credibility* is in essence the validity of the study, ensuring that the main focus of the study was in fact studied. *Dependability* focuses on the methodology used in the study to make claims and generalisations which would be acceptable in the broader research community. *Confirmability* focuses on the researcher's influence and bias of the study. In order to ensure the confirmability and dependability of a study, limitations, the position of the researcher and ethical requirements will need to be disclosed in full.

An in depth description of the details of the case study, including the limitations, assists in ensuring the transferability and credibility of the study, by showing the reality of the case. In this study, the data were checked by external researchers to ensure that it was interpreted in similar ways (Creswell, 2014). Dependability was ensured by explaining each method used in the collection of the data and validating why it was appropriate.

Creswell (2014) believes that strategies such as triangulation, clarifying bias and external auditing can be also used to ensure the validity of a study. Thomas (2011) defines triangulation as looking at a study from different angles. In this study, data collection methods included learners' test papers (written work), semi-structured interviews and focus group interview recordings. Using this variety of methods ensured the topic was 'fleshed out' and therefore added a degree of authenticity (Hopkins, 2014). The responses from the learners during the focus group activities added depth to the evidence from the written tests, which highlighted the learners' level of algebraic thinking.

3.7 Role of the researcher

Bias is described by Olsen (2012) as "an essentially one-sided viewpoint or specifically grounded standpoint on a phenomenon". These types of bias are present in many situations and can therefore be an expected part of research. The researcher herself can bring an aspect of bias to the research study (Olsen, 2012). Creswell (2014) adds that the researcher should clarify the bias she adds to the study, which could be shaped by her background, experience and culture. Having taught in Grade 1-3 using both curricula, the researcher's personal opinions about them could have resulted in the pre-empting of discussion topics in this study. The analysis process may therefore have been subject to bias due to this. Care was taken when collecting the data using a variety of instruments, in an attempt to limit this bias. The researcher was unknown to both sample groups and was therefore not tempted to favour one group over the other. Researcher bias was lessened by the fact that the researcher was not fulfilling the role of both the teacher and the researcher. In this study, the researcher aimed to maintain an open mind set and to be critical of all comments and observations made during the data collection process (Olsen, 2012). Entry to both schools was gained by obtaining permission from the 'gatekeepers', including the parents of the learners involved, class teachers, Heads of Department and school principals (Creswell, 2014).

3.8 Ethical considerations

Ethical consideration was adhered to by following the three principles of research ethics requirements, namely *autonomy, non-maleficence* and *beneficence* (Rule & John, 2011).

Autonomy involves the researcher's responsibility to respect and protect all participants involved in the study and to respect their right to withdraw from the study at any point. For this study, consent forms explaining the study were sent to the teachers from both schools, along with a permission letter to each principal. As the sample consisted of minors, consent forms were sent to the parents of the minors asking their permission to participate in the study. The aim of the study was outlined and the participants given the option to withdraw from the study at any point, if they so wished. Consent forms were received before the interviews began. As both schools used in this study were independent schools, consent forms from the Western Cape Education Department (WCED) were not necessary. Permission was granted by the Cape Peninsula University of Technology. These permission letters can be found in Appendix 3-7.

Non-maleficence ensures the safety of all participants and their communities throughout the study. For this study, the learners were never left unattended and were supervised throughout the study by either their class teacher or by the researcher and an assistant.

Beneficence suggests that the results of the study be beneficial and contribute to the public good. For this study, the principals, teachers and parents were informed that the results of the study will be made available for their perusal. Feedback and staff workshops will be offered to school principals on completion of the study.

3.9 Conclusion

The aim of this study was to understand how two curricula provide for the development of algebraic thinking in the Foundation Phase. A case study was conducted and the data collected using a variety of collection methods. This data were analysed using the frameworks developed by Tarr *et al*. (2006) and Blanton *et al*. (2015). This analysis process allowed for patterns to emerge within the data which provided a deeper understanding of these specific cases, including the opportunities for and evidence of functional thinking. Reliability, validity and trustworthiness was maintained throughout the study. The research project was conducted ethically with permission from all stakeholders granted before the commencement of the study.

CHAPTER 4

FINDINGS AND DISCUSSION

4.1 Introduction

This chapter presents the findings of this study into the development of algebraic thinking as related to the research questions provided in the chapter. Different data collection methods were used to understand how two different curricula provide for the development of algebraic thinking in the Foundation Phase The findings from the study are organised in two parts, namely Part A and Part B. Part A focuses on how the curriculum documents, textbooks and teachers, who are responsible for the delivery of the content, approach the development of algebraic thinking. Part B relates to how the learners perform in a test of algebraic thinking and the results of focus group task-based interviews which were used to explore their functional thinking. The chapter concludes with an overview of the findings from Part A and Part B in relation to the aim of the research.

4.2 Part A

4.2.1 Curriculum and Textbooks

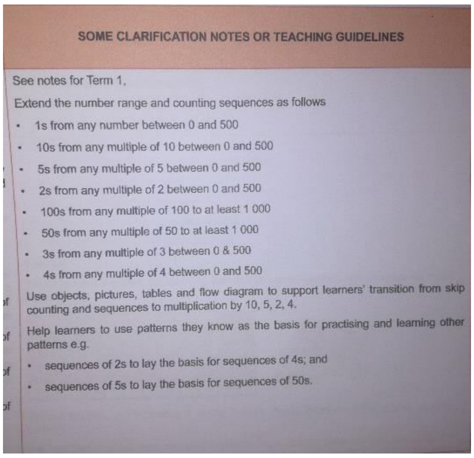
How do the CAPS and SM curricula and textbooks provide for the development of algebraic thinking in the Foundation Phase?

Curriculum documents and textbooks facilitate the acquisition of knowledge in classrooms (Garner, 1992). The analysis of these documents (i) clarifies what is included in the two mathematics curricula and how this is translated and presented to the learners, and (ii) assists in understanding the overall development of algebraic thinking in the Foundation Phase in the South African context. This section of the chapter explains the findings of each curriculum and textbook; and the content relating to patterns, functions and algebra. Selected examples from the CAPS and SM Mathematics textbooks are reviewed in order to understand how each curriculum has been translated into the classroom. The selected examples are then linked to the theoretical framework to ascertain the levels of the examples used. A discussion of these examples is included at the end of the section. The framework developed by Tarr *et al*. (2006) for analysing textbooks was used as the analytical tool. This framework is divided into: *mathematical content analysis, instructional focus and teacher support* as explained in methodology in Chapter Three.

4.2.1.1 CAPS curriculum

The CAPS curriculum is made up of several mathematical strands related to content. One of the strands relates to patterns, functions and algebra. In the Foundation Phase, learners are taught about geometric and number patterns. They are expected to complete and extend geometric and number patterns, as well as create and describe their own patterns. The Department of Education (SA.DBE) (2011:365) states that the learners must “copy, extend and describe in words simple patterns made with physical objects…” Copying a pattern assists the learners in seeing the logic of how a pattern is created, while extending patterns helps them ensure they have understood this logic. Describing patterns assists in the development of the learners’ language skills. The focus on the logic of patterns forms the foundation for the development of algebraic thinking skills. Geometric patterns include lines, shapes, objects and patterns in the world. When working with these patterns, learners apply their knowledge of shape and space. The learners work with growing geometric patterns where the elements are repeated in the same way. The size of the shape changes and the number of shapes or objects changes in a predictable way.

Number patterns are included in this strand and the curriculum states that the learners are expected to copy, extend and describe number sequences within an appropriate number range, as seen below in Figure 4.1.



SOME CLARIFICATION NOTES OR TEACHING GUIDELINES

See notes for Term 1.

Extend the number range and counting sequences as follows

- 1s from any number between 0 and 500
- 10s from any multiple of 10 between 0 and 500
- 5s from any multiple of 5 between 0 and 500
- 2s from any multiple of 2 between 0 and 500
- 100s from any multiple of 100 to at least 1 000
- 50s from any multiple of 50 to at least 1 000
- 3s from any multiple of 3 between 0 & 500
- 4s from any multiple of 4 between 0 and 500

Use objects, pictures, tables and flow diagram to support learners' transition from skip counting and sequences to multiplication by 10, 5, 2, 4.

Help learners to use patterns they know as the basis for practising and learning other patterns e.g.

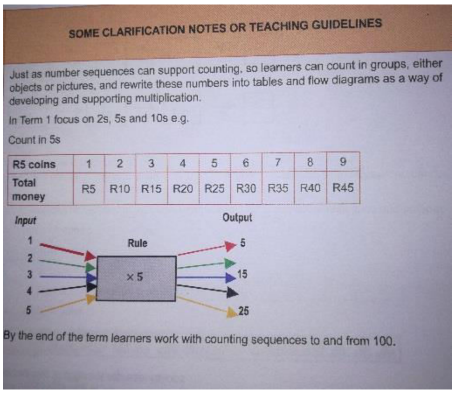
- sequences of 2s to lay the basis for sequences of 4s; and
- sequences of 5s to lay the basis for sequences of 50s.

Figure 4.1: Clarification notes or teaching guidelines (SA.DBE, 2011:431)

Number patterns support the concepts of numbers and operations built into Numbers, Operations and Relationships. The curriculum document encourages the use of pictures, tables and flow diagrams (function machines) in the pattern section.

The SA.DBE document (2011:477) explains that the aim of this is to "support learners' transition from skip counting and sequences to multiplication by 10, 5, 2 and 4."

The CAPS curriculum document includes a clarification section where each item is explained with examples. This clarification section provides insight into how patterns are taught. The learners are taught about repeating geometric patterns and patterns found in the world around them, including symmetry. Different models are used for teaching geometric and number patterns, including 2D and 3D shapes, tables, number lines, number charts and number grids. Initially, they are offered guiding questions to assist them in describing the patterns, but these are no longer used by Grade 3. The number patterns are closely linked to counting and this link is made explicit. Number lines are used to identify patterns and learners are expected to find missing numbers in a sequence or in a number line, table or chart. The models are introduced as the grades progress and the number range steadily increases. An example of the clarification for number patterns in Term 1 of Grade 3 is shown below in Figure 4.2.


SOME CLARIFICATION NOTES OR TEACHING GUIDELINES

Just as number sequences can support counting, so learners can count in groups, either objects or pictures, and rewrite these numbers into tables and flow diagrams as a way of developing and supporting multiplication.

In Term 1 focus on 2s, 5s and 10s e.g.

Count in 5s

R5 coins	1	2	3	4	5	6	7	8	9
Total money	R5	R10	R15	R20	R25	R30	R35	R40	R45

By the end of the term learners work with counting sequences to and from 100.

Figure 4.2: Clarification notes or teaching guidelines (SA.DBE, 2011:368)

4.2.1.2 SM curriculum

The Singapore mathematics curriculum for the primary years is organised into three content strands, with mathematical processes that cut across the strands. The processes are defined as the skills involved in learning and acquiring mathematical knowledge. These include *reasoning, communication and connections, applications and thinking skills and heuristics* (Singapore. Ministry of Education, 2012:15). These processes are not taught deliberately, but through the teaching and learning of concepts and skills. In the primary curriculum, learners are exposed to both geometric and number patterns. The geometric patterns fall under the sub-section of geometry. In this section, learners are expected to create, complete and describe patterns made of 2D and 3D shapes according to specific attributes (size, colour, shape and orientation). In Primary 2 (Grade 2) under the strand of geometry, the curriculum states that "students should have the opportunity to make/complete patterns with 3D shapes (except sphere) according to one or two attributes…and explain the pattern" (Singapore. Ministry of Education, 2012:41). Number patterns are covered from Primary 1 through Primary 3 (Grades 1-3) and begins with learners "describing patterns using 1 more, 1 less and 10 more, 10 less before continuing the pattern or finding the missing number(s)" (Singapore. Ministry of Education, 2012:34). By the end of Primary 3 (Grade 3), learners are expected to describe number patterns and continue the pattern by filling in missing numbers. The number range increases each year. The curriculum document includes learning experiences that accompany each strand and sub-strand within it. These learning experiences are defined by the Singapore. Ministry of Education (2012:20) as "experiences…to influence the ways teachers teach and students learn so that the curriculum objectives can be achieved". They give examples of what the learners should do when guided correctly by the teacher and focus on the processes and skills related to the topic. The learning experiences are specific enough to ensure guidance is correctly provided whilst still trying to give the teachers room for flexibility. An example of a learning experience for a pattern activity in Primary 2 (Grade 2) states that "students should have opportunities to describe a given number pattern before continuing the pattern or finding the missing number(s)" (2012:37).

4.2.1.3 CAPS textbooks

For this study, the researcher analysed the textbooks used by the sample schools. The CAPS school used Simply Maths (Cawood, 2006) and New Wave Mental Maths (Krajcar, 2014). In terms of geometric patterns, the activities in the CAPS textbooks require learners to extend the patterns by drawing the next shape or picture in the sequence. Some examples give a few options of what the next picture would be and the learners had to choose the correct answer, as shown in Figure 4.3 below.

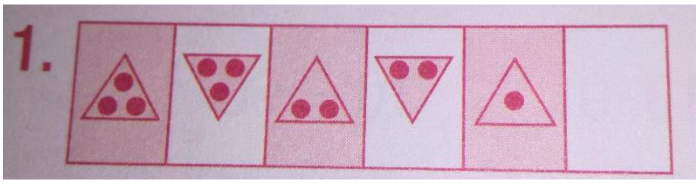


Figure 4.3: Example of shape sequence (Krajcar, 2014:4)

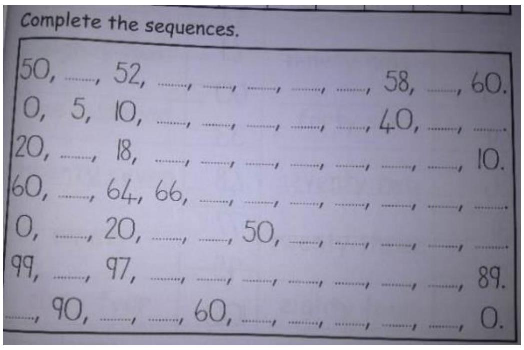
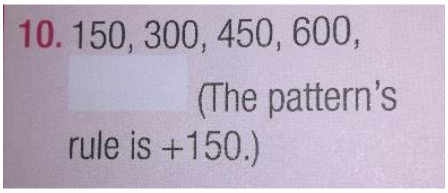
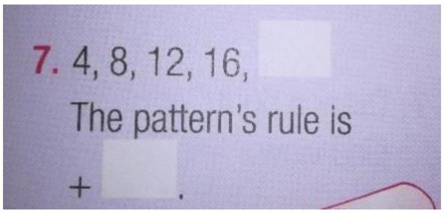
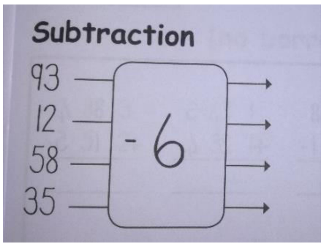
Table 4.1 below shows selected examples from the CAPS textbook. These have been linked to the levels of sophistication in describing children's functional thinking as developed by Blanton *et al*. (2015). This linkage determines which levels these examples fall into. In keeping with the curriculum requirements, most of the examples below are recursive in nature. According to the Blanton *et al.* (2015: 529) framework used, these levels are defined as follows:

Level 2 Recursive Particular (L2–RP): Learner conceptualises a recursive pattern but does not see it as a generalisation. Learners' understanding of the relationship is still represented as a counting process. An example of this is when a learner explains a pattern in a data table as a particular sequence of values (e.g. "two, four, six, eight")

Level 3 Recursive General (L3-RG): The pattern is seen as a recursive pattern, however, the learner now has a general rule for that pattern. This is similar to the previous level but the learner can now explain that the next value will be 2 more than the last. An example of this level is when a learner explains that "you must add 2 more cars to the number of cars you have now"

The following table provides four different number pattern activities found in the selected CAPS compliant textbooks.

Table 4.1: Activities found in CAPS textbooks (Cawood, 2006: 1-11 & Krajcar, 2014: 64-65)

Activity	Level	Link
Counting patterns – learners finish sequence by filling in numbers in sequence. Counting forwards in 5's, 4's, 2's, 50's. Backwards in 2's, 5's, 10's, 100's.  	Level 2 Recursive Particular	The emphasis of this question is on counting. This example demonstrates that the next value is dependent on the previous level. The number sequences have varying starting points and only consist of 2 digit numbers. There are a variety of counting sequences.
Number sequences where the learners have to complete the sequence by filling in the next value. The pattern rule is given.  	Level 3 Recursive General	The emphasis of this example is on sequences and relationships. By providing the rule, it demonstrates a recursive relationship. It focuses on the common difference or recursion and not on the relationship between the terms.
Number sequences where the learners have to describe the pattern by providing the rule.  	Level 3 Recursive General	The emphasis of this example is on describing patterns. The structure of the question indicates that the focus is on the recursive nature of the sequence.
Function table where inputs and function rules are given. The input values appear random. The learners are expected to find the function values (outputs).  	Level 2 Recursive Particular, but related to functional thinking as the learners work across the data.	In this example, the representation changes. This representation demonstrates more emphasis on the relationships between the terms, showing a move towards functional thinking. Due to the fact that the function rule is provided, the activity remains on level 2 as a generalisation has not been made.

The number sequence activities selected from the CAPS textbooks appear to match the aims of the curriculum and include a variety of representations of patterns. The first example requires learners to analyse the given sequences to find the missing numbers, both positive and negative. The second example provides the rule and shows no focus on predictions or how values relate. Counting using a rule prepares the learners for working with pattern rules. Counting sequences which require learners to provide a rule when given the sign, prepare them for working with rules which focus on recursion. The function machines (flow diagram) demonstrates a shift where the emphasis is on the relationship between the input and output values. The examples require the learners to provide the output values when the inputs are given. No examples including finding the function rule (generalising) or finding missing input values were found in either of the CAPS textbooks.

4.2.1.4 SM textbooks

The textbook series used in this study is My Pals Are Here (Kheong, Ramakrishnan & Choo, 2007). In this textbook, the algebra strand is translated as "order and pattern" and is a sub section under the main heading of "Numbers to 1000". The SM textbook start with geometric pattern activities by providing the learners with a few shape patterns and asking the learners to identify this pattern, as seen in figure 4.4 below.

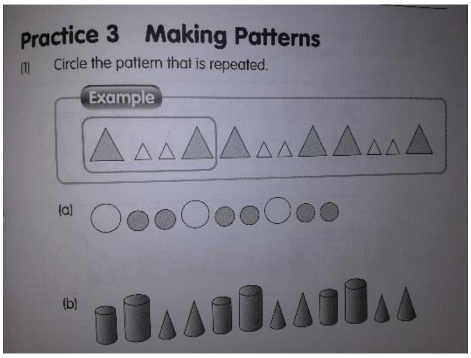
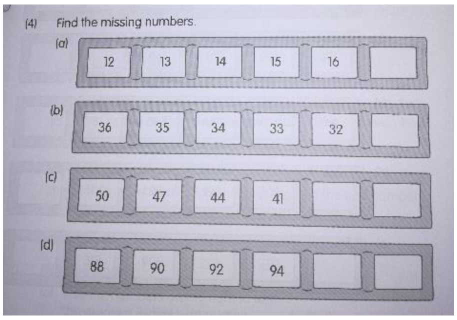
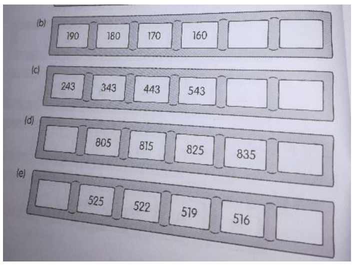
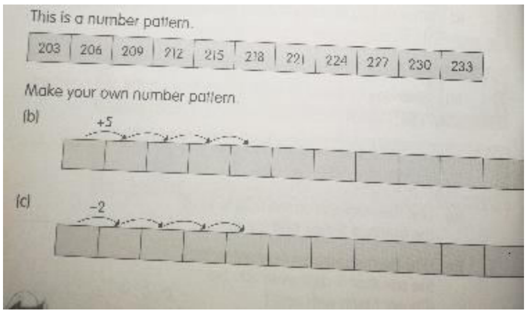
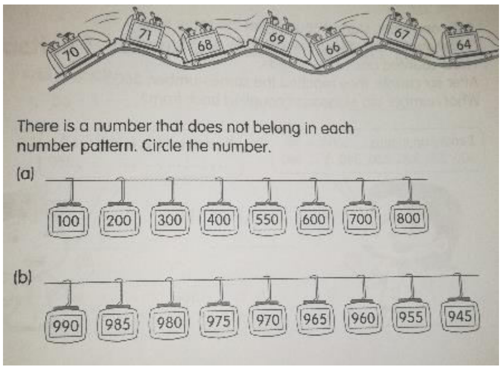


Figure 4.4: Example of Making Patterns (Kheong *et al*, 2014a:91)

The table below shows examples from the SM textbook and how they link to Blanton *et al*.'s (2015) levels of sophistication. This linking determines which levels these examples fall into. In keeping with the curriculum requirements, most of the examples below are recursive in nature.

Table 4.2: Activities found in SM textbook (Kheong et al, 2014b: 15-21)

Activity	Level	Link
Counting patterns – learners finish sequence by filling in numbers in sequence. Counting forwards in 1's and 2's. Backwards in 1's and 3's.  	Level 2 Recursive Particular	The emphasis of this activity is on counting as the learners are expected to continue the pattern. It demonstrates that the next value is dependent on the previous one. The counting pattern of the sequences is a combination of addition and subtraction and the sequences are made up of 2 digit numbers.
Counting patterns – learners missing numbers. Counting forwards and backwards in 10's, 100's and 3's.  	Level 2 Recursive Particular	Counting is the focus of this activity. The number range is larger and the activity is represented as a counting pattern. The first pattern subtracts 10 each time using decade numbers. The next pattern counts back in 100's showing a link between 10's and 100's.
Learners complete patterns given the rule and starting number  	Level 3 Recursive General	Learners continue the counting pattern by following the rule given. This is a counting activity, but in a different form. The requirement is slightly different as the learners now apply the rule to create their own pattern.
Learners asked to describe number patterns and find the term that does not fit.  	Level 3 Recursive General	This activity is recursive in nature but includes opportunities for the learners to analyse patterns. This analysis had to be done in order to find the term that did not belong. A different representation to patterns than seen previously.

The activities selected from the SM textbook include a variety of number pattern activities. The number patterns found in the SM textbook show a series of numbers where the learners must analyse the pattern in order to provide the missing values. The number ranges increase from 2 digits to 3 digits and have differences of 1, 3 and 10, both positive and negative. Examples where learners must continue the pattern when provided with a rule demonstrate a different approach to patterns and prepares them for working with patterns. When asked to create their own pattern, the learners apply the recursive rule. A different set of thinking skills is applied in the example which requires learners to find the value that does not belong in the sequence, as the learners must analyse the pattern. This example demonstrates a shift toward using counting pattern to apply different skills, as opposed to just counting.

4.2.1.5 Discussion of curricula and textbooks

In this section, the curriculum documents and textbooks from CAPS and SM will be discussed using the framework and categories developed by Tarr *et al*. (2006). In this discussion, the categories: *instructional focus* and *teacher support* will be merged, as both the curriculum documents and textbooks provide guidelines for the teacher.

a) Mathematical content analysis:

Mathematical content analysis refers to how well a textbook supports the expectations of a curriculum. This includes the development of skills, conceptual understanding and the development of mathematical processes. In terms of the curricula, both CAPS and SM provide a detailed description of what is expected. They both include knowledge and skills which learners should acquire through their mathematics lessons. The SM curriculum includes deeper explanations of these processes and what they mean.

Geometric patterns are used differently in each curriculum and textbook. While the CAPS curriculum includes both number and geometric patterns in the algebra strand, in the SM curriculum they fall under geometry. Both the CAPS and the SM curricula focuses on copying and extending patterns using geometric shapes.

The activities found in both textbooks support what is expected from their respective curricula. The activities focus generally on the skill of extending patterns, which is an expectation of both CAPS and SM. The textbooks provide good linkage and appear to be an adequate translation of the expectations of the curricula. Although presented differently, the number patterns in both textbooks are similar and the focus is on counting. The examples from the textbooks show number patterns playing a critical role in the algebra strand, although neither textbook shows evidence of a focus on functional relationships. Although there are activities that lend itself to this, the questions and representations that promote this way of thinking are not evident in either textbook function machine where the rule is provided is an arithmetic task, as the learners have to calculate the output values. In order to shift this to a functional thinking focus, the structure of the question would need to change. Some input and output values can be provided, and the learners could find the function rule. In doing this, the functional relationships between the input and output values are more emphasised. An example in the SM textbook also required a different set of skills as the learners had to analyse the pattern in order to find the value that did not belong in the sequence.

Both these examples highlight that the CAPS and SM curricula and textbooks have the opportunity to develop functional thinking. Blanton *et al*. (2015) state that young children are able to generalise the functional relationship between co-varying quantities and with further support and well-designed tasks, their levels of understanding could improve and become more sophisticated. Carraher *et al*. (2006) agree, emphasising that the correct activities could bring mathematical ideas together that may have otherwise remained isolated, resulting in a deeper conceptual understanding.

b) Instructional focus and teacher support

Instructional focus looks at how a textbook's activities develop mathematics as a "way of thinking" and looks into whether the textbooks are being used as a teaching and learning tool. *Teacher support* focuses on whether the materials provide teachers with the opportunity to increase their own knowledge and understanding of the content.

There is support and guidance for teachers in both the CAPS and SM curriculum documents. The CAPS documents refers to this in the clarification section. This clarification explains the expected outcomes of each section. As an example, the clarification for a pattern activity includes: "*extend the number range and counting sequences as follows: 1's from any number between 0 and 500, 50's from any multiple of 50 to at least 100. Help learners to use patterns they know as the basis for practising and learning other patterns e.g. sequences of 5's to lay the basis for* *sequences of 50's".* The SM curriculum refers to this support as learning experiences. These learning experiences provide an explanation of what is expected and guide the teachers into achieving the aims of the curriculum. The support related to working with number patterns is similar in both curriculum documents. Both documents explain that the learners must extend and describe the patterns.

A difference in terms of support is the teacher's guide used alongside the SM textbook. This guide provides a detailed explanation of how the intended outcomes of the lesson, the main concepts, are to be taught as well as the thinking skills that are to be demonstrated. In a number pattern activity, the teacher's guide explains as follows: "*get pupils to try to see the pattern using the words 'more than and less than'…pupils should notice that the difference between the numbers is 10 and not 1 as before".* The thinking skills for this number pattern lesson included "*identifying relationships"* (Kheong *et al*., 2007:21).

Blanton and Kaput (2011) explain that in order for learners to think algebraically, their thinking needs to be scaffolded and guided. Guidance and scaffolding is provided in each curriculum document. The guidance towards developing functional thinking is not evident, however. Both curricula documents and textbooks seem to focus on counting and multiplicative reasoning to explain and solve pattern problems. Although the documents refer to relationships and using patterns as the basis for other patterns, the focus is not on finding a functional relationship between the terms. There are too few examples and opportunities provided to work with functional relationships. This is critical if we hope to develop a learner's algebraic thinking (Blanton *et el*., 2015).

4.2.2 Sub-question 2: Teacher interviews

What are the teachers' perceptions of the development of algebraic thinking within each curriculum?

The teachers' perceptions have been included in the research as they are an important aspect in the shaping of the curriculum and serve as the link between the curriculum and the child's reality (Remillard & Bryans, 2004). The aim of the interview was to understand how the teachers use and interpret the curriculum and textbooks to develop their learners' algebraic thinking. It included questions related to planning and teaching of pattern lessons. When discussing curricula, it is important to note the different purposes of curricula. These are *intended, enacted, assessed* and *learned*.

For this discussion, the focus is on the *enacted* curriculum, which is defined as the actual curricular content that the learners are taught in their daily instruction. It includes the content taught and the manner in which it is taught. Most learning occurs within the enacted curriculum and this information provides a deeper understanding of how the learners in this study were taught and what content was covered.

The teachers were interviewed using semi-structured interviews. To understand the teachers' perceptions of patterns and to ascertain what and how they taught their learners, they were asked questions related to tasks involving algebraic thinking. This refers to the way of thinking about functional relationships between terms and involves the building, expressing and justifying of mathematical relationships using pattern activities. This is addressed in mathematical practices in the theoretical framework section of Chapter Two.

4.2.2.1 CAPS teacher

The CAPS teacher explained that many of her pattern activities are found in the textbooks she uses in her class. When working with patterns, she explained that she did mostly number patterns where the learners had to count forwards and backwards. She referred to these patterns as "repetitive" patterns. She explained that while teaching these, she would draw her learners' attention to the position of the digits. When counting in 2's, she would write the numbers underneath each other to illustrate how the patterns look the same:

Interviewer: How do you teach patterns?

Teacher: Ok so, number patterns. I usually write them on the board. So when introducing a new counting pattern, obviously they've done this in Grades 1 and 2. In Grade 3 we just remind them, then count with them on the board. I try to write it so that you can visually see the number pattern. So when you counting in the tens it ends in a certain digit and when counting in 20s it ends in a certain digit.

Teacher writes: 0 2 4 6 8 10

10 12 14 16 18 20

When doing number pattern activities in their maths groups, her learners wrote out a number pattern on a white board, erased numbers from the sequence and swopped with a partner. The partner then completed the missing sequence. This teacher revelead that she thinks patterns are an important part of mathematics as it helps and encourages the learners to look for patterns in numbers. The teacher explained that "*pattern work is about having an organised mathematical understanding and being able to put things into patterns*". The teacher thought that doing pattern activities influenced the learners' mathematics journeys as it caused them to "*think in a different way*".

4.2.2.2 SM teacher

The SM teacher interviewed for the purposes of this study also uses the pattern activities from the textbook and teachers guide which guides her planning and teaching. As the SM system is rather prescriptive, the teacher explained that she follows the lessons and activities prescribed in the textbooks. When asked how she teaches patterns, the teacher explained that each new concept is introduced using an anchor lesson. This introductory lesson is taught to the class and the content and questions are provided in the teacher's guide. The teacher's guide provides the objectives of the activity, the thinking skills to be covered within that lesson and an instructional procedure. The instructional procedure gives clear steps to follow when teaching the lesson, including the vocabulary to use. It includes what the learners should do or take note of during the discussion.

In her interview, this teacher explained that during pattern activities she makes the links between bigger and smaller numbers explicit:

> *"So we would count in 3's and then what would happen if we count in 30's?"*

This teacher thought that patterns had been included into the maths curriculum, because Singapore Mathematics is about "*understanding the breakdown of the numbers*" and that patterns help learners gain an understanding of number. She explained that the patterns taught in the Singapore Maths curriculum are not focused on rote learning and counting, but instead is more about recognising relationships between numbers such as 2 x 3 and 20 x 30.

4.2.2.3 Discussion of teacher interviews

From their interviews, both teachers seemed to understand patterns as a sequence of numbers or objects that can be extended and described. They both demonstrated an understanding of patterns being closely linked to multiplicative reasoning. With the main focus on extension, the learners seem to be working between levels 2

(Recursive Particular) and 3 (Recursive General) of the levels of sophistication (Blanton et al. 2015). Both teachers used their respective curriculum documents and textbooks to guide their planning and teaching of patterns. One distinct difference between the two teachers, is that the SM teacher had a teacher's guide which the CAPS teacher did not. This support guides her teaching and includes questions to ask her learners to ensure their learning is scaffolded. The CAPS teacher did not have this support included in her textbook.

Blanton and Kaput (2004) found that recursive patterning is included in mathematics classroom activities. Mathematics activities in the younger grades, however, often omit co-variation and correspondence relationships, or the opportunities for this. Blanton *et al*. (2015) found that learners in younger grades do not require extensive experiences with recursive patterns in order to think of functional relationships. Instead, the learners in their study demonstrated an ability to think about functional relationships whilst performing poorly on recursive pattern questions. This explicit way of thinking requires learners to establish relationships between terms and their positions in the sequence. From here the learners can begin describing the rule in their own words.

The teacher covering the Singapore Maths curriculum thought that pattern activities assist learners in understanding numbers as well as assisting them in visualising what would come next. She further explains her understanding when providing the example of 2x3 and 20x30 and how she makes these relationships explicit, drawing her learners' attention to these types of relationships.

The example from the CAPS teacher where the learners complete number patterns on the whiteboards encourages the learners to think recursively and to count on. Although this way of thinking is important, there is little evidence of moving the learners towards looking for relationships

The CAPS teacher mentioned that pattern activities are important as they help the learners to "think in a different way". Mulligan, Cavanagh and Keanan-Brown (2012) agree, explaining that Early Algebra is not only thinking about algebra from an earlier age, but includes re-looking at numbers from a different perspective. The teacher appears to be helping her learners and suggests that to think algebraically and find these patterns, her learners need to be '*thinking in a different way*'.

The examples given by the teachers in their interviews do not seem to go beyond the third level of the levels of sophistication, recursive general (Blanton *et al*., 2015). Up to this level, the learners are not yet focused on finding relationships within patterns or describing patterns in their own words. The learners are able to continue a sequence of numbers and can explain that the next value will be ‘2 more’ than the previous one. Blanton and Kaput (2011:9) believe that elementary school learners have the capacity for functional thinking and found that the representations learners use, the development of their mathematical language used when describing relationships, the methods they use to organise data and their ability to express co-varying and corresponding relationships can be “scaffolded in instruction” from very early on in their formal school career.

The teachers were both working with pattern activities with a focus on growing the pattern. Some activities encouraged learners to look for relationships within patterns, but highlighted that both teachers understand patterns as being closely linked to multiplicative reasoning. In both situations, there was very little reference in terms of predicting beyond what was given, forcing the learners to think more in order to find relationships between the terms, such as functional thinking. Most of the activities completed by both of the teachers have the potential to push the learners beyond arithmetic and into a more functional way of thinking where they can identify relationships between numbers (Blanton & Kaput, 2011). Although these opportunities were present, the potential to use them in the development of functional thinking was not mentioned by either teacher.

These missed opportunities could be attributed to the teachers’ limited pedagogical knowledge of functional thinking. The curriculum states that learners must extend and describe patterns, which the teachers do, according to their enacted curricula. During the teaching and learning of these activities, there is no evidence of patterns moving beyond the specialisation and towards creating generalisations.

4.2.3 Summary of Part A

In terms of curriculum documents and textbooks, both include pattern activities within their respective algebra strand. The textbooks from each curriculum provide activities that will assist learners in achieving the aims of their specific curriculum and the teachers from both groups are fulfilling the requirements as stated in their respective curriculum. There is evidence from both teachers that they are beginning to make the links between counting patterns and multiplicative reasoning, but there is no evidence of this moving towards learners finding relationships and creating generalisations (functional thinking). Although this is not yet a requirement in either curriculum, Blanton *et al*. (2015) explain that young learners have the potential to make connections and, with the correct scaffolding, could begin to represent these connections in more symbolic ways. The textbooks provide examples that could be adapted and used to promote functional thinking. Professional development, however, would be required in order to make teachers aware of how to guide and scaffold the development of functional thinking, using the tasks they use on a daily basis.

4.3 Part B

4.3.1 Sub-question 3: Learner pencil and paper test

How do learners using CAPS and SM curricula solve pattern problems?

Two different instruments were used to ascertain learners' algebraic thinking. This was to gain a deeper understanding of how the learners use what they are taught and the examples from their textbooks to answer algebraic thinking type questions. Firstly, a pencil and paper test was designed and completed by all learners from both classes. The test was designed using examples from past ANA question papers as well as taking examples from textbooks at the Grade 3 level. The test was made up of nine questions which covered a variety of items relating to both generalised arithmetic and functional thinking, as mentioned in Chapter 3 at paragraph 3.1. This was carried out in an attempt to gain an overall understanding of the learners' algebraic thinking. Secondly, learners were interviewed in a mixed ability group made up of eight learners from each class to probe more deeply into their algebraic thinking. The groups were seen on different days and the researcher facilitated by asking questions and encouraging discussions. The learners were given two functional thinking problems to solve and were audio recorded while they discussed and solved the problems.

This section will highlight the findings from each group using both instruments, namely the pencil and paper test and the focus group interviews. These findings will be discussed individually and later summarised together. (Refer to Appendix 1)

Below is a table showing each group's correct percentage performance on the items found in the paper test.

Table 4.3: Percentage performance on paper test

Question and content	Singapore Maths School correct percentage	CAPS school correct percentage
Q1: Number counting pattern	84	96
Q2: Number counting pattern	88	86
Q3: Number counting pattern	64	93
Q4: Equivalence	60	75
Q5: Equivalence	32	44
Q6: Working with a variable	56	75
Q7: Function table	36	41
Q8: Geometric pattern	48	68
Q9: Function rule	40	86
Average	56%	73%

According to the data provided in the above table, the CAPS learners performed better overall on this test. The results indicate that both groups were successful in completing Questions 1 and 2, the number pattern questions related to addition and subtraction. Both groups were challenged by the equivalence question, Questions 4 and 5, when asked to find the missing number to make an equation true, i.e. "6 + 9 = ☐ + 10" or "18 + 9 + 16 = ☐ + 12". The counting pattern in Question 3 required the learners to see the relationship between counting in fours and counting in forties. There was a large discrepancy between the results in this question, with the CAPS learners achieving significantly higher results. Question 6 required the learners to work with variables. The results indicate that, although using a trial and error method, many learners were able to correctly find the variable with a higher success rate from the CAPS group. The function table in Question 7 produced low success levels from both groups. Question 8 was a geometric pattern where sticks were used to build squares. The results from this question had a large discrepancy between the two groups, with the CAPS group achieving higher results. Question 9 was a function machine where learners had to provide either the input or output value. The data indicates a large discrepancy in these results, with the CAPS group achieving higher levels of success. The most common difficulties among both sets of learners was calculating the input value when provided with the output, and calculating the output with the larger input value.

This findings section focuses on Questions 3, 7 and 8 and classifies learners' methods used to answer these questions. Questions 1, 2, 4, 5 and 6 were not included in a deeper analysis process. Both groups scored well in Questions 1, 2 and 4 and therefore a deeper analysis was not necessary. Question 5 produced low levels of success from both groups, but was not a functional thinking question. Question 6 worked with a variable and most learners were able to find the correct answer, with many using a trial and error method. Algebraic thinking, for the purpose of this study, is defined in terms of functional thinking, so these types of questions were chosen for analysis. Question 9 had the largest discrepancy between the results but was not chosen for discussion. The nature of that question could be used to determine functional thinking, but the way in which it was presented caused it to focus on arithmetic and not algebra. By providing the learners with the function rule, their functional thinking abilities were no longer being assessed. Instead, the aim of the question became to calculate and provide an answer. For this reason, Question 9 was not selected for further discussion. Question 3 was selected as it is the type of question the learners would see on a regular basis. The focus in the question is on describing the relation between the patterns and the large discrepancy between the results; which makes it a point of interest and worth discussing. The low levels of success from both groups in Question 7 provides the reason for this question being selected for further discussion. The geometric pattern in Question 8 is a different type of question to what the learners see in their textbooks. The learners were asked to predict a near and far term and to explain their rule, in other words algebraic thinking. This question provided a large discrepancy in the results, and will be discussed in more detail.

4.3.1.1 CAPS results

a. Question 3: Number pattern counting in fours and forties

The CAPS learners appear to perform better overall in this question with a success rate of 96%. They were able to explain the relationship between the two patterns. Many learners from the CAPS group answered, as shown in the below figure.

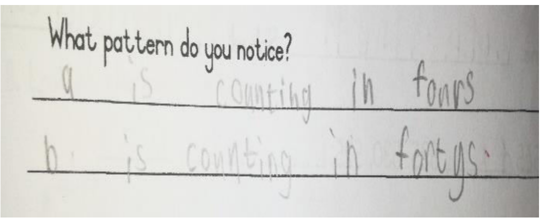


Figure 4.5: Example answer from Question 3

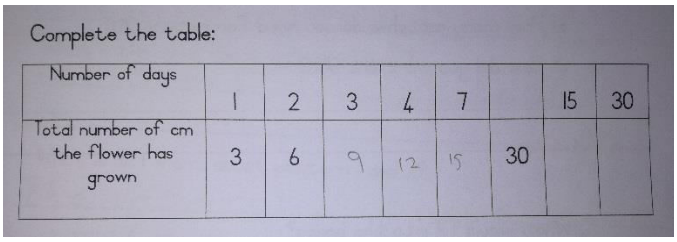
The learner above explains that one pattern counts in fours while the other counts in forties. This type of response falls into level 2, recursive particular, of Blanton *et al*.'s (2015) levels of sophistication. In this level, the learners conceptualise a recursive pattern as a sequence of particular instances. The pattern is seen as a counting process and not as a relationship. While most of the responses were similar, some learners moved beyond this level when answering this question. Another learner explained that she noticed the following when looking at the two number patterns:

"I found out on pattern b you copy pattern a and add a nought to the end"

This type of response indicates that the learner appears to show the beginning of linking knowledge, although the learner could not fully explain the mathematics transformation taking place. She appears to see the patterns as a relationship, but does not explain why the zero is added onto the end. According to the levels of sophistication this learner is operating at level 3, recursive general (RG) (Blanton *et al*., 2015). In this level, the learners' primary description reflects a general rule. They are able to conceptualise a recursive pattern as a general rule between arbitrary values, but make no reference to a particular instance. In this example, the general rule reflected is that a zero will always be added. The learner does not explain that the second pattern is ten times larger than the first one.

b. Question 7: Function table with linear equation y=ax

This question is functional in nature as it requires the learners to find a relationship between the terms, predict a far term and explain how this was achieved. Of the CAPS learners, 41% were able to correctly complete the function table, using different methods. Four examples are included showing the different methods. Below is an example of a learner (Learner 1) who was unable to complete the table.



Complete the table:

Number of days	1	2	3	4	7		15	30
Total number of cm the flower has grown	3	6	9	12	15	30		

Figure 4.6: Example answer from Question 7 from Learner 1

The example appears to demonstrate that this learner is operating on the second (recursive particular) of Blanton *et al.*'s levels of sophistication (2015:525-539). The terms that have been filled in seem to indicate that the learner views the pattern as a counting sequence, but there is no indication to show whether she is adding three each time or using a rote counting approach. Rote counting in 3's would be a level 2, recursive particular response, while adding 3 each time would fall within level 3, recursive general. When the input value changed from 4 to 7, learner 1 continued adding 3, which could indicate that the focus was not on the relationship between the input and output values.

When predicting the far term, how many days the flower would take to reach 150cm, 41% of the CAPS learners gave the correct answer. There were different explanations given by learners as they used different methods. Learner 2 counted in three's and explained this as her method to predicting the far term. This method falls into level 2, Recursive Particular, of Blanton *et al.*'s levels of sophistication (2015:525-539), as she conceptualises the recursive pattern as a counting process and only uses the output values. Learner 3 explained her thinking as follows:

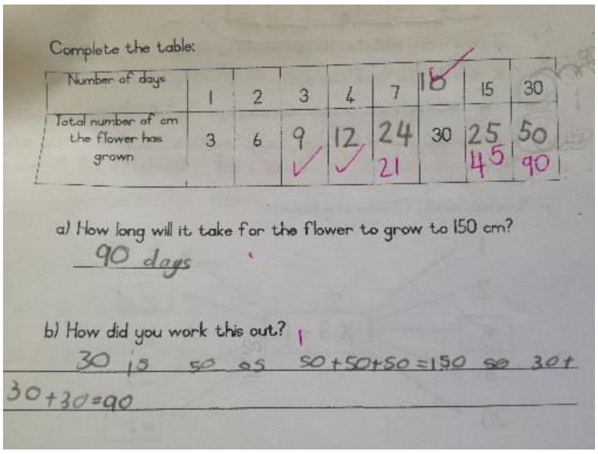


Figure 4.7: Example answer from Learner 3

Learner 3 used multiplicative reasoning to predict the far term. She multiplied her output value until she reached the output in the prediction question (her output in the function table was incorrect). She found that the output height was three times her last output amount. She then applied this to her input amount to get the number of days. This example appears to demonstrate that Learner 3 may have some understanding of relationships. In terms of Blanton *et al.*'s levels of sophistication (2015:525-539), this is classified as a Level 3 (recursive general) response. This learner created a general rule, but uses it in a recursive way. She applied her rule to both terms, but moved between the values instead of across the values (from input to output).

Another response to this question was to use the inverse operation to predict the far term. After completing the table and correctly predicting the number of days the flower would need to reach 150cm, some learners, for example, explained that they divided 150 by 3. This response demonstrates that some learners are able to recognise the relationship between the input and output as they were able to use the reverse operations. According to the Blanton levels, this would be a level 6 (EFG) response. In this level, the learners show attributes of a generalised, functional relationship, however, their representation of this relationship is not entirely complete. In this example, the relationship is not explained in words although the mathematical transformation is explained. Another example of a level 6 (EFG) response is shown below.

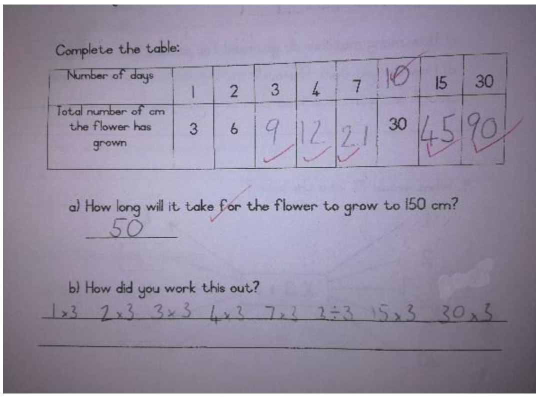


Figure 4.8: Example of a level 6 response

Learner 4 used his knowledge of the three times table to answer the prediction question and create a rule. By doing this, he shows the emergence of a generalised, functional relationship as he sees the generality of the relationship between the terms. His representation is incomplete as he does not explain in words how the terms relate to one another. As with the previous example, Learner 4 is able use inverse operations.

c. Question 8: Geometric pattern with linear equation y=ax+b

This question was a geometric pattern of squares made using sticks and was answered using a variety of responses. These responses show different levels on which the learners operate and will be explained further. The learners were given the first three pictures in the sequence and were asked to draw picture 4 and 6. They were asked to predict how many sticks they would need for a far term (picture 20) and had to justify their answer by explaining how they achieved this. With regards to the CAPS learners 68% were able to draw pictures 4 and 6 correctly. Predicting the far term was challenging and only 6% of this group answered it correctly. The responses for how they worked this out varied. The first method used by some learners was to draw the 20 triangles and count each stick. The learners who chose this method are operating at level 2 (RP) of Blanton *et al.*'s levels of sophistication (2015:525-539). These learners conceptualised this pattern as a counting process where they counted each stick individually between terms. Another learner, operating at level 3 (RG) used the following method:

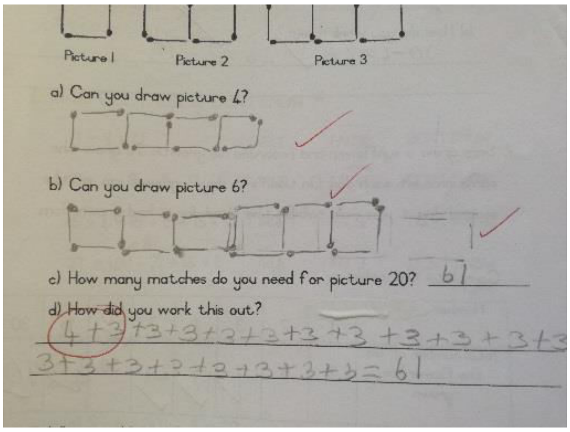


Figure 4.9: Example answer from Question 3

This learner calculated accurately and her explanation of what she did reflects a general rule, i.e. 3 is added each time. She has conceptualised a recursive pattern as a generalised rule between arbitrary values with no reference to particular instances.

Other learners used the knowledge that the first square used 4 sticks, so multiplied 4 by 20 to find the number of sticks needed for 20 squares. This method shows the use of multiplicative reasoning and demonstrates that this learner is generalising from a single incidence, but does not show evidence that she recognises the structure of the question. In terms of the levels of sophistication (Blanton *et al.*), this method would be a level 3 (recursive general) response, even though the answer was incorrect. This method demonstrates some understanding of relationships, but not of the nature of the growing pattern. A description of this level is that the learner conceptualises a recursive pattern as a generalised rule and is able to use this rule to find specific values.

A different method, as seen below, was used by another learner and she explained her thinking as follows:

19x3=27+4=31

This method demonstrates this learner's understanding and ability to think algebraically, although she made a calculation error. Her methods shows that she understands the nature of the growing pattern as she multiplied 3 sticks by 19 (for 19 squares). She then added 4 (sticks) for the first square. This method falls into level 4, functional particular, according to the Blanton levels. A description of this level is that the learner is able to calculate y when x is known, as the functional relationship is conceptualised as a set of particular relationships between certain, corresponding values.

4.3.1.2 SM results

a. Question 3: Number pattern counting in fours and forties

Of the three number patterns in the pencil paper test, the SM learners found Question 3 the most challenging. The responses from the learners varied and fell within different levels of the Blanton *et al.* levels of sophistication (2015). Some learners in this group answered the question as follows:

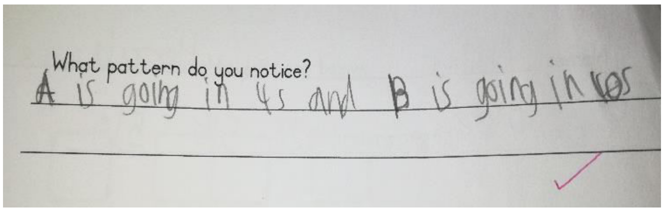


Figure 4.10: Example answer from Question 3

This learner's response is within level 2 (recursive particular) of the levels of sophistication. At this level, the learners conceptualise the recursive pattern as a counting pattern and not a general relationship. The learner sees that each pattern is increasing and noticed the amount that each term grows.

Another learner explained that she noticed the following:

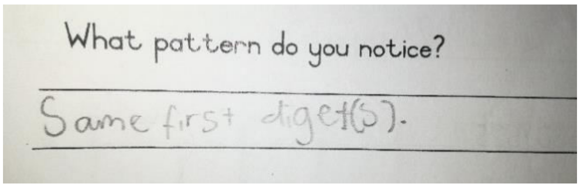


Figure 4.11: Example answer from Question 3

From this explanation, it seems as though the learner has seen that the second pattern has a zero as the last digit of every number. This learner's explanation demonstrates that she noticed a pattern, however, does not provide further evidence to show her understanding of how these patterns relate to one another.

b. Question 7: Function table with linear equation y=ax

In this question, which was related to the growth of a sunflower, the learners were required to find a relationship between the terms in order to predict a far term. In the SM group, 36% of learners were able to correctly complete the data table. Many of these learners were operating on level 2, RP, of Blanton *et al*.'s levels of sophistication (2015:525-539). Below are two examples of learners' work:

Learner 1 Learner 2

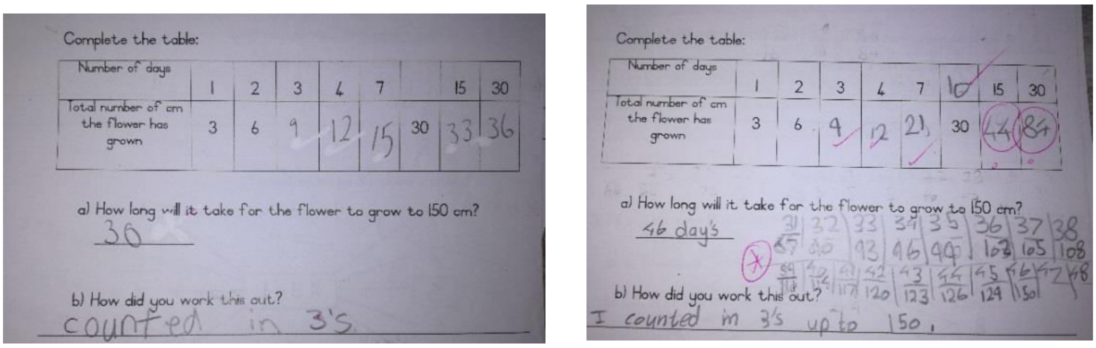


Figure 4.12: Example answers from Question 7

Learner 1 and 2's work falls within this level because they seem to have understood the relationship as a recursive pattern and see it as a counting process. Neither one has conceptualised it as a generalised relationship. Although both incorrect, the methods used by these learners are different and cause them to fall in different levels of sophistication. Learner A counts in three's along the output values of the data table, irrespective of the input value changing. He does not appear to look for links between the terms. Learner B also counts in three's but seems to demonstrate more of an understanding that input and output terms could be related in some way. He looks for links between the terms by adding on 3 each time. Conceptualising a recursive pattern as a generalised rule is a description of Blanton's third level, recursive general. Making the link between the input and output values demonstrates an ability to think algebraically.

Another learner working within level 2, recursive particular, of the levels of sophistication (Blanton *et al.*, 2015) also sees the relationship as a counting pattern but uses arrows to indicate movement across the data instead of along inputs and outputs, as seen below in Figure 1.14.

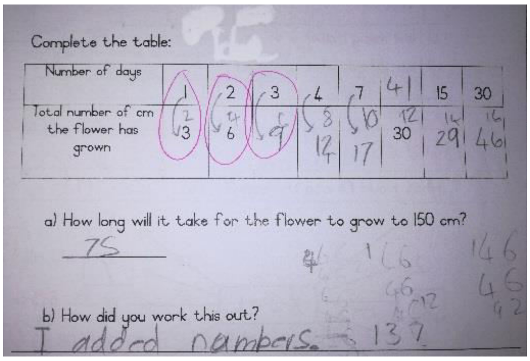


Figure 4.13: Example answer from Question 7

In this example, we see the learner making a link between the number of days and the height of the flower. This learner sees the relationship as addition and then finds a recursive pattern between the inputs and outputs. When looking at this example one might think the learner is beyond level 2, recursive particular, but upon closer inspection it is clear that he has still conceptualised the pattern as a counting process in relation to the number of days and the total growth of the sunflower. This example is interesting because it demonstrates that this learner is able to see that values can relate to each other in a functional relationship.

Some learners in this group explained that they would divide 150 by 3 to find out how long the flower would take to reach 150cm. This type of response shows their ability to understand the relationship as being three times the amount of days. Their explanations, however, were incomplete as they were unable to explain it. One learner explained her thinking as seen below.

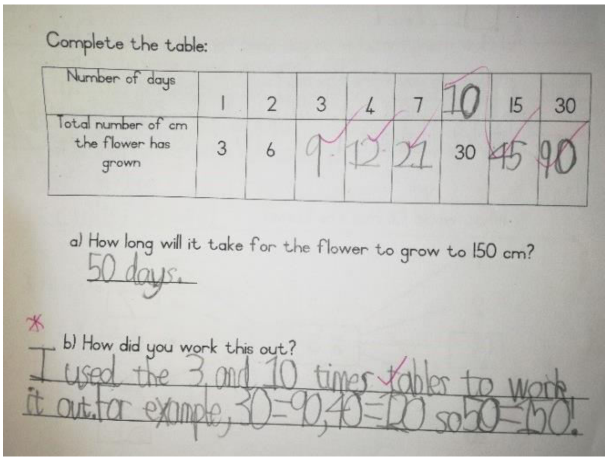


Figure 4.14: Example answer from Question 7

This example could be a level 6 (EFG) response according to Blanton *et al*.'s levels of sophistication (2015:525-539). In this level, the learner begins to highlight key attributes of a generalised, functional relationship. The example demonstrates this learner's understanding of how terms can relate to each other, but does not show that he is able to describe the relationship in words or symbols.

c. Question 8: Geometric pattern with linear equation y=ax+b

Question 8 required the learners to extend and predict a geometric pattern represented using sticks to build squares. They were provided with the first 3 pictures in the sequence and were asked to draw picture 4 and 6. They were then asked to predict how many sticks would be required to build a far term (picture 20). When drawing pictures 4 and 6, 48% of the SM group were able to draw them correctly. Predicting the far term was challenging, with only 8% of this group able to correctly predict the number of sticks needed for picture 20.

The methods used by this group were similar to those used by the CAPS learners. Most of the SM learners used either a draw and count method where they drew picture 20 and counted the sticks. This method demonstrates the learners conceptualising the pattern as a counting process, making this response a level 2, recursive particular, response according to Blanton *et al*.'s levels of sophistication (2015:525-539). Other learners working in the same level explained their thinking using natural language, as seen in the example below.

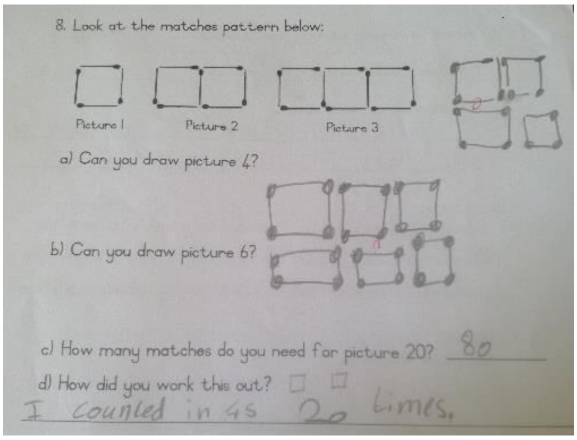


Figure 4.15: Example answer from Question 8

The learners working within level 3, RG, of Blanton *et al*.'s levels of sophistication (2015:525-539) explained that they were adding three sticks on for every new square built. One learner explained his thinking as follows:

"*Pic(ture) 1 has 4 matches. Pic(ture)2 has 7 so I added 3 for each number for 20".*

By explaining that he added 3 each time, this learner has conceptualised the recursive pattern as a generalised rule, but with no references to particular instances. He relied on the previous value in order to predict further.

Some learners in this group used previous knowledge to incorrectly predict the far term. They used the number of sticks needed for 10 triangles and doubled it to get the number of sticks needed for 20 triangles. This demonstrates their use of multiplicative reasoning skills. It does not show any evidence of their ability to see relationships between terms and demonstrates an over generalisation.

Another learner from the SM group recorded his information by writing the number of squares and the number of sticks needed below each term. This appears to represent a data table. The bottom of the picture shows the input values with the output values written underneath each one. He was able to answer the prediction question correctly. As seen in the picture below, he drew the 20 squares first and then transferred the information into his data table.

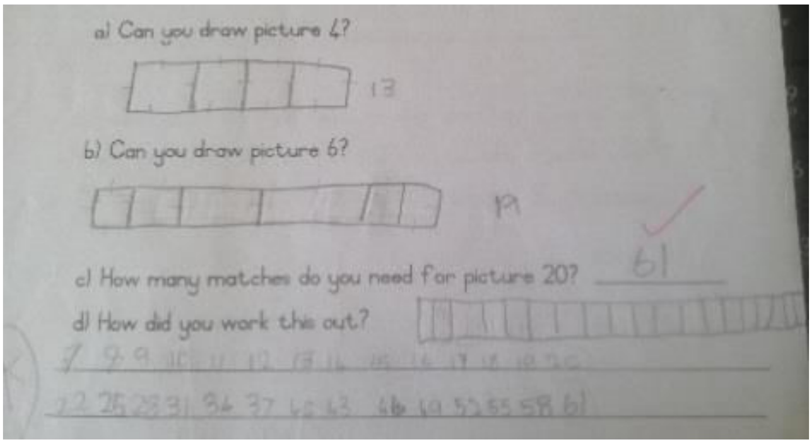


Figure 4.16: Example answer from Question 8

Although there is no evidence that he didn't simply count the sticks and insert the information into his function table, this examples highlights a young learner's ability to recognise the link between different representations of functions.

4.3.1.3 Discussion of pencil and paper test

The CAPS learners performed better overall in this test, as shown in Table 4.3. As seen in the findings, the methods used by the learners in both groups were often very similar. Blanton *et al*. (2015) explain that the levels of sophistication are not hierarchical and that children need not pass through every level in order to progress to the next. Instead, they noticed during their study that some children skipped levels while others moved through multiple stages during one question. The CAPS and SM learners displayed the same results as both groups were operating on a variety of levels. This demonstrates the different levels of understanding of functional relationships and algebraic thinking.

Patterns and functions are included by Greenes (2004) as one of the leading ideas of algebra. Greenes explains that growing and shrinking patterns help lead children to make generalisations and later represent those using variables. Learners in both the CAPS and the SM group were able to make some form of generalisation about the growing pattern in Question 8. Although there was no evidence that they could explain the relationship, some learners demonstrated their understanding through their calculations, for example, the learner who wrote *19x3=(27)+4= (31).* Despite the calculation error, the generalisation that every square after the first one uses 3 sticks is evident. An SM learner demonstrated his understanding by recording his information using a data table. This shows what he understands about terms relating to one another in a functional way. Many learners from both groups were operating on level 2, recursive particular, where they saw the pattern as a counting sequence and not a generalised relationship. Both groups had different learners operating on level 3 and 4, in which they conceptualised a general rule or demonstrated an understanding thereof. The learners in level 3, recursive general, were not able to reference a particular instance and their primary reflection still revealed a general rule in a particular instance. These learners could explain that they added 3 sticks for every square, but could not provide a rule. There were very few learners functioning on level 4, functional particular, and they could describe the rule in a specific case, but not as a generalised, functional relationship.

When answering the counting patterns in Question 3 and the data table in Question 7, many learners saw the recursive nature of the patterns. Twohill (2016) explains that this type of thinking is instinctive for children and with scaffolding and questioning they can begin to think explicitly (functional thinking) about patterns. The learners in both the CAPS and SM groups displayed evidence of this. Some CAPS learners were operating on level 3, recursive general, of the levels of sophistication (Blanton *et al*., 2015) where they began to conceptualise a recursive pattern as a generalised rule but with no reference to the particular problem. Some learners in the SM group displayed evidence of functional thinking where they attempted to explain the relationship between the growth of the flower and the number of days it took to grow.

4.3.2 Focus group interviews

A mixed ability group of eight learners per group was chosen from each class to participate in focus group interviews. During these interviews, the learners were given two pattern functional thinking tasks to solve. The tasks represent different function relationships and the learners were encouraged to work with a partner and explain their thinking as they worked. The first problem was ‘eyes on dogs’ and it focused on the relationship between the number of dogs and the total number of eyes on the dogs (assuming each dog has 2 eyes). The function type addressed in this question was $y = x+x$ or $y=2x$. The second question was also a growing pattern where the learners used sticks to build a row of triangles, each time adding one more triangle to the existing triangle pattern. The focus was on the relationship between the number of triangles and the number of sticks needed to create them. The function type addressed in this question was $y = x+x+1$ or $y=2x+1$. These questions were selected as the functions appear in this order in the sequence of tasks used by Blanton *et al*. (2015) and their

functions are different in structure. These questions were used in the focus group interviews to ascertain how the learners would cope and solve them. The focus is on the way the learners think about the problems and the methods they use to solve them. The interviews were audio recorded in order to assist with the data analysis process. The questions were read to the learners and they were encouraged to discuss their thinking with a partner. During this process, I observed them while they discussed and worked through the problems, asking questions where necessary.
The two questions were presented as follows:

a. Question 1: Dogs and eyes

I went to a park and saw lots of dogs. I wanted to know how many eyes there were altogether. Do you think you could help me work it out? First I saw 2 dogs, how many eyes? 4 dogs? 6 dogs? Then I saw 10 dogs. How can I work this out? Must I count every dog's eyes? What if there were 20 dogs?

Do you think we could make up a rule that will help us to always find the number of eyes?

b. Question 2: Triangles and sticks

Look at the triangles we have built using the sticks. With a friend, can you answer the following questions:

How many sticks do we need to make 3 triangles?

How many sticks do we need to make 5 triangles?

How many sticks do we need to make 10 triangles?

What if I wanted to make 20 triangles? Is there a rule we can use?

4.3.2.1 CAPS Results

a. Dogs and eyes problem

The learners in this group were able to answer the first few questions verbally, as seen below:

Interviewer: Okay so that dog had 2 eyes. Then I looked around and I saw 2 dogs…how many eyes now?

Learner 5: Four!

Interviewer: Good. Then I looked again and I saw 4 dogs…how many eyes were there now?

Learner 2: Eight!

They required assistance when constructing a vertical data table, but were able to do so and complete it correctly. When asked about a rule one could use to predict a further term, most learners in this group answered that one must count in twos along the data table. This response indicates that these learners were operating within level 2, recursive particular, of Blanton *et al*.'s levels of sophistication (2015:525-539).

Others, working on a different level, explained one could multiply by 2 to find the function values (output). When asked how many eyes there would be on 8 dogs, this learner answered correctly and explained that she "*timesed 2 by 8"*. One learner had the same idea, however confused her quantities:

Interviewer: Do you think we could make up a rule? To always work out how many eyes there would be?

Learner 3: Times 2

Interviewer: Times what by 2?

Learner 2: Times the dogs eyes by 2

Interviewer: The dogs eyes by 2?

Learner 5: No, times the dogs by 2.

Interviewer: Okay, but why?

Learner 6: Well 2 times is double, so you could just double.

Learner 5: You do that because 2 eyes is on 1 dog

Both learners in the above example were showing some understanding of how quantities can relate to one another. Although they were able to explain their rule, their explanations are incomplete. The first learner could verbalise that she multiplied 8 by 2 and included the mathematics in her explanation (8x2) of the relationship, but had not included the quantities used. The other learner, with scaffolding, included one quantity

(2) and the mathematics used in the relationship (multiplication), but had not explained the rule entirely using both quantities and the mathematics taking place. Both these responses fall within level 6, EFG, of Blanton *et al*.'s levels of sophistication (2015:525-539). In this level, the learners' thinking reflects the beginning of key attributes of a generalised, functional relationship.

c. Triangles and sticks problem

The second question was the growing pattern related to triangles built out of sticks. The first three pictures in the sequence were given to the learners, as well as a few sticks to extend the pattern, if necessary. They were asked to build the first few pictures and then had to provide the number of sticks needed for 3, 5, 10 and 20 triangles, and asked to explain their thinking.

When asked to build the first 1, 2 and 3 triangles using the sticks, the learners were easily able to do this. They correctly built the first one and many learners mentioned that they added sticks to continue the pattern, as seen below:

Figure 4.17: Learners building triangles with sticks

They were asked to predict how many sticks they would need for 5 and 10 triangles. The researcher watched and listened to their discussions and asked questions while they worked. The responses fell within various levels of the Blanton *et al.* levels of sophistication (2015). One group explained that they noticed a pattern in the sequence of triangles. When asked what the pattern was, they explained that the triangles were facing different directions. One was facing up, then the next facing down. This response is indicative of a level 1, pre-structural response. In this level, the learners do not describe any mathematical relationship when discussing, but may notice something non-mathematical.

Some learners in this group were operating within level 3, recursive general, of the levels of sophistication (Blanton *et al*., 2015). When asked how many sticks they would need to build 8 triangles, the following conversation took place:

Learner 6: When you have 1 triangle you plus 3, but then for the next you plus 2 and plus 2 and plus 2

Learner 5: You start off with 3, then keep plusing 2 each time (wrote the number sentence)

Interviewer: How many times are you going to plus 2?

*Learner 5: *counts* 7 times*

Interviewer: Why 7?

*Learner 5: Because you start with 1 triangle *points to 3* and add 7 more (triangles) to make 8*

Another learner in a different pair, when explaining her rule, said "*you start off with 3 and always plus 2*". By using the word "*always"*, this learner shows her understanding of a generalised rule that can occur between values. In this level, RG, the learners' primary description begins to reflect a general rule, as seen in both examples above. Another learner, also working at this level, created a data table to record her information.

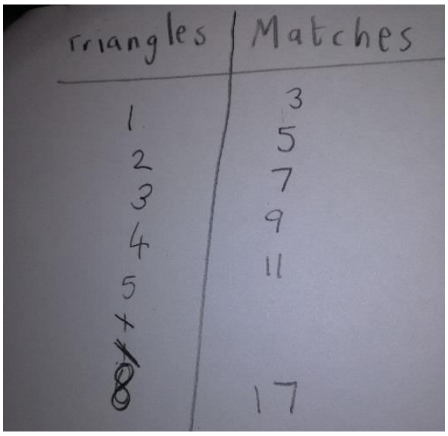


Figure 4.18: Learner's table to predict number of sticks

As seen above, this learner had to add pictures 6 and 7 in order to predict the further, picture 8. This implies that this learner is adding 2 down the output column. A characteristic of this level is that the learner relies on the previous output value in order to determine the next one and is working at a recursive general level.

A different group of learners in the interview relied on previous data in order to predict a further term. When asked how many sticks would be required to build 10 triangles, they explained as follows:

Learner: 30 sticks

Interviewer: Why?

Learner: We said 10 times 3

Interviewer: Why 3?

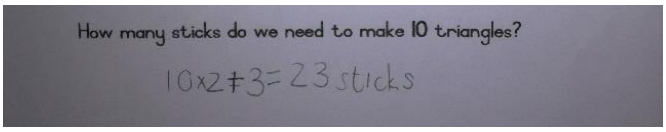
Learner: Because you need 3 matches to make 1 triangle

Interviewer: So how many sticks did you use to make 2 triangles? Six?

Learner: Oh…no…it's not right.

These learners use multiplicative reasoning incorrectly to predict a far term. Although they appear to see a link between the number of triangles and the number of sticks needed, their explanation suggests that they generalised the particular and did not relate the problem to the structure of the problem.

Another group seemed to be working on level 6, emergent functional general, of the levels of sophistication. This group completed a data table and seemed to look "across" the table. When asked what the "rule" could be, one learner explained that they doubled the number of triangles and then added 3. They explained that they added 3 because the first triangle used 3 sticks. They tested their rule against their data table and found it was unsuccessful. They then used a 'trial and error' method and tried x2+2 as the rule, which was also unsuccessful. Finally they decided to double and add 1, which provided the correct output results.



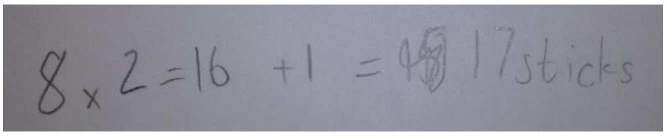


Figure 4.19: Learners' answers to predict number of sticks for 10 triangles

This response was classified as a level 6 response, because, although initially incorrect, these learners showed the emergence of key attributes of a generalised, functional relationship. Although they explained the mathematical transformation between the quantities, the representation of the relationship was incomplete. They had not yet used both quantities when describing their relationship.

4.3.2.2 SM results

a. Dogs and eyes problem

The learners in this group were able to answer the first few questions verbally and, when encouraged, were able to construct and complete a data table. The first few questions required the learners to record how many eyes there would be on 2, 3 and then 4 dogs. The next question was how many eyes there would be on 6 dogs. When answering this question, one learner completed his data table as follows:

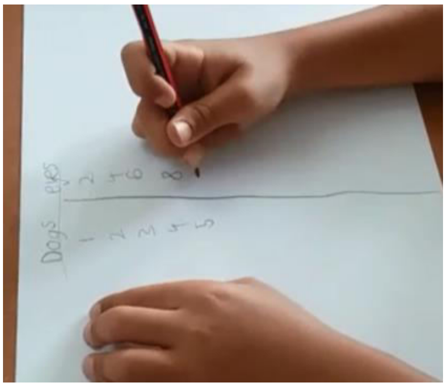


Figure 4.20: Learners' table to predict number of sticks for 10 triangles

In order to find the number of eyes on 6 dogs, the learner above first needed to calculate the eyes on 5 dogs which he added into his data table. This method highlights that this learner is operating within the recursive particular and recursive general levels of the Blanton *et al*. (2015) levels of sophistication. The learner has not yet conceptualised the recursive pattern as a general rule, but rather sees it as a counting pattern where the output values follow a counting sequence. One cannot determine whether the learner was counting in two's (level 2, recursive particular) or adding two each time (level 3, recursive general). It is clear, however, that this learner had not yet developed an understanding of how to explain how quantities can be related to each other, despite him having the correct answers.

Another pair of learners, working together, were operating on different levels, namely level 2 (recursive particular) and level 5 (primitive functional general). When asked for a rule we could use to always find the number of eyes on any amount of dogs, the following conversation took place:

Learner 1: *It's counting in twos*

Learner 2: *No, its times by 2*

Interviewer: *Times what by 2?*

Learner 3: *The dogs*

Interviewer: *Why?*

From this example, it seems as though learner 1 was operating in level 2, RP. His explanation of counting in two's shows that he has conceptualised a recursive pattern as a string of particular sequences and not as a generalised relationship. He has seen the output values as a counting pattern. His partner, learner 2, correctly explains that he multiplied one quantity by 2 in order to find the other. Although he has explained the mathematical operation between the quantities, he could not articulate a relationship that explains the quantities being compared i.e. the relationship between then number of dogs and the number of eyes. Learner 3 tries to assist when scaffolded by the interviewer.

Another learner was asked why she thought she had to double the number of dogs. She explained:

> *"Because each dog has 2 eyes so you must double the number of dogs to get the eyes"*

Although prompted, this learner shows evidence of working within level 6, Emergent Functional General, of the levels of sophistication. In her explanation, she shows her understanding of key attributes of a generalised, functional relationship.

When asked how many dogs there were if I saw 30 eyes, 2 learners impulsively shouted 60 dogs, then quickly changed their minds. One learner counted on her fingers and told me the correct answer. When asked how she got her answer, she explained:

> *"I know that there was 30 eyes so I counted in twos up till 30 and then you would see how much dogs you would get"*

This was asked to ascertain if these learners would use the inverse operation to find the number of dogs. Instead, the explanation above demonstrates that this learner reverted to counting in order to calculate the number of dogs, making this a recursive particular response.

b. Triangles and sticks problem

When presented with this problem, some learners in this group used the sticks to build the first few pictures. Many of them created a data table to record their information and discussed their thinking in pairs. They were all able to answer the first few questions orally, using the pictures they had built. When asked to predict a further term, term 20, a variety of methods were explored.

The most common explanation was to add 2 to the previous output value. One learner explained it as follows:

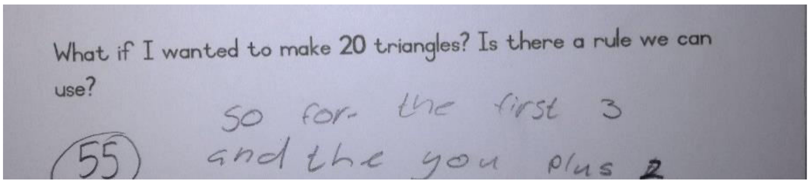


Figure 4.21: Learner's solution to 20 triangles

The method used here falls into level 3, recursive general, of the Blanton *et al*. (2015) levels of sophistication. In order to predict a far term, they would need the previous term, to keep adding 2.

Another popular method was to use previous knowledge in order to predict a further term. Some learners in this group explained that they would multiply by 3, because each triangle is made up of 3 sticks. Similarly, another pair said they would double the sticks used for picture 10 in order to find the number of sticks needed for picture 20. As seen in the CAPS results, these learners also used multiplicative reasoning to predict the far term. This demonstrates that they did not understand the relationship between the terms and did not move towards functional thinking. Another learner and her partner disagreed when predicting how many sticks would be needed to build 3 triangles. Learner 4 was using multiplicative reasoning to predict how many sticks would be required, while her partner seemed to display an understanding of the structure of the pattern. The following conversation took place:

> *Learner 4: That's so easy, its 9 (sticks)*
>
> *Learner 5: No it isn't! You already have the first stick then you must add 2 each time.*
>
> *Interviewer: What are you discussing here?*

Learner 5: Well people would think it's 9 (sticks) because you need 3 sticks for 1 triangle. So you keep adding 3, but it isn't. You start with 3 (sticks) then keep adding 2 each time because 1 stick is already there.

Although Learner 5 has not yet conceptualised the pattern as a generalised relationship, her explanation demonstrates that she is beginning to understand the concept of a general rule where she says "*keep adding 2 each time".* This response is within the recursive general level of Blanton's levels.

Learners who used the number of sticks needed for picture 5 answer to predict the number of sticks needed for picture 10 (by doubling) were asked to build picture 10 to test their method. After building it, they found that they needed 21 sticks instead of the 22 they had recorded. This confused them and the following conversation took place:

Interviewer: So how do we get from 10 triangles to 21 sticks?

Learner 6: Well, we tried to times it by 2 but you can't times it by 2 because that would be 20.

Learner 7: It's kind of a thing, there's a pattern here.

Interviewer: And what is that pattern?

Learner 7: The pattern is that there is going to be a 1 each time. But you can times 20 by 2 and then times the 1 by 2….umm….

This state of confusion forced this pair to look deeper at their data in an attempt to find a relationship between the number of triangles and the number of sticks needed to build them. Their conversation demonstrates the beginning of an understanding of a functional relationship between the number of triangles and the number of sticks required. It seems, however, as though they were over-generalising and using a trial and error method. The exercise of building the picture to test their theory was important, as it caused them to rethink their method and could be used as an opportunity by the teacher to move the learners towards looking for a functional relationship.

After building the pictures, one learner recorded her information on paper as follows:

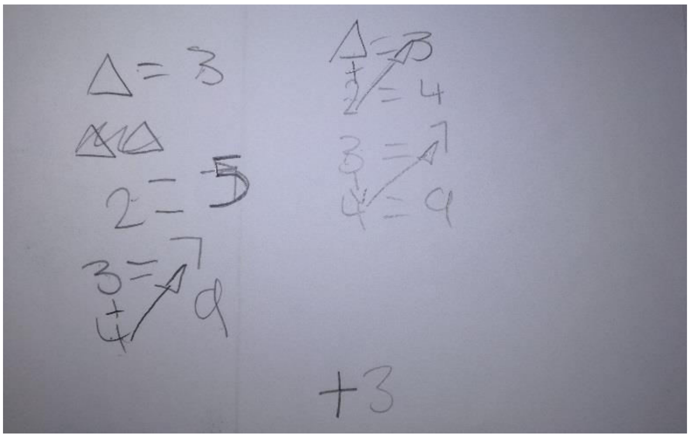


Figure 4.22: Learner's solution to triangles and sticks question

From her data, we see that this learner was trying to find a relationship between the number of triangles and the number of sticks needed to build them. The arrows indicate that she is seeing a pattern by adding 2 input values and getting the output as the answer. After working through it for some time, this learner explained her findings as follows:

> *Interviewer: Can you tell me what you found?*
>
> *Learner 3: Yes, well it will always be double and then minus 1.*
>
> *Interviewer: Why minus 1?*
>
> *Learner 3: Because you had 1 (stick) in the beginning.*

This response demonstrates this learner is beginning to understand how quantities can relate to each other. By adding an equal sign between the number of triangles and the number of sticks required indicates an understanding of the problem. Her arrows creating the pattern of adding 2 input values to get an output suggests a trial and error approach. Her explanation of "*always"* seems to suggest her understanding that a rule can be generalised, even though her rule was incorrect.

4.3.2.3 Discussion of focus group interviews

Both CAPS and SM learners coped well with the question relating to the dogs and eyes. They were both able to explain a functional relationship between the number of dogs and the number of eyes. Furthermore, they both explained this relationship using natural, spoken language (not symbols) which Blanton and Kaput (2011) explain is a normal starting point. Some learners required some guidance and scaffolding when trying to explain the relationship, but Blanton and Kaput (2011) explain that although children have the capacity to reason about functions and use tools to represent this, they need well designed activities and support from the teacher.

The learners from both groups also demonstrated their ability to move between Blanton's levels of sophistication. This indicates the various levels of understanding of functional thinking present in these groups. Moreover, both groups of learners used similar methods to answer the questions in the triangles and sticks problem. The similarities included the majority of both groups operating in levels 2 (recursive particular) and 3 (recursive general) of the Blanton *et al*. (2015) levels of sophistication. In these levels, the recursive nature of the pattern is the focus while the learners attempt to find a relationship between the quantities. Twohill (2016) explains that this is the natural starting point for children when finding patterns and relationships. Learners from both groups displayed their potential to develop functional thinking abilities using multiplicative reasoning to predict a further term. Many learners in the CAPS group used this approach. The researcher realised later that the learners should have been asked to build the picture to test their answers.

A pair of learners in the CAPS group decided to build the far term and found their rule to double picture 5's answer to get picture 10's did not work. Learners in the SM group also used this multiplicative reasoning approach and after testing it, found it unsuccessful. The learners in the CAPS interview saw their rule was incorrect and used a trial and error method to find the rule that produced the correct output. When asked why she added 1 after doubling the number of triangles, one learner responded: *"because if you did anything else you would not get the right answers".* Learners in the SM group also tried to find the rule to produce the output value. Although the SM learner mentioned the one stick that is "extra", she was not able to correctly identify the function rule for the question.

Learners from both groups demonstrated a different way of thinking than what they demonstrated during the pencil and paper test. The function table question in the pencil and paper test produced low levels of success for both the CAPS and SM groups. The majority of learners remained in the recursive particular level where the emphasis was on counting. Some moved into the recursive general level where they were able to conceptualise a general rule, but used it in a recursive way. When completing the dogs and eyes activity, which was of a similar function structure ($y=2x$ and $y=3x$), most of the learners were more able to see the relationship between the terms and could explain that the number of eyes is double the number of dogs.

The scaffolding and guidance during the focus group activities forced some learners to think about the problem in a different way and brought them closer to finding a functional relationship.

4.3.3 Summary of Part B

The findings of the pencil and paper test and the focus group interviews relate to recent research which found that young children are able to think algebraically, given correct and sufficient guidance and scaffolding (Blanton & Kaput, 2011). The learners' ability to see a functional relationship and understand how terms relate to each other, demonstrates that these learners have the potential to be guided towards algebraic development. The questioning and scaffolding during the focus group interviews demonstrates how specific questioning can make learners rethink their method and force them to approach a problem differently. The movement across Blanton's levels of sophistication highlights the potential and the different levels of understanding of relationships that is already present among the learners. The teacher plays an important role in the planning and sequencing of activities. Carraher *et al*. (2006) explains that teachers need to find the opportunities within mathematics tasks to develop the learners' algebraic thinking skills and must redesign tasks if necessary.

4.4 Conclusion

From the findings, it is evident learners have the potential to think algebraically. The curriculum documents and textbooks provide guidelines and activities related to algebraic thinking, although the focus remains on recursive patterning. The teachers in this study followed these documents and guided their learners through the activities. It is clear that the teachers identify a close link between patterns and multiplicative reasoning and encouraged this way of thinking among their learners. The learners' ability to record data and begin to explain how the terms relate to one another showed they have the potential to develop a function rule, as defined by Moss and London McNab (2011). The learners from both groups moved between the different levels of sophistication of thinking (Blanton *et al*., 2015) depending on the nature of the function, demonstrating the different levels of understanding present amongst the learners. Some relied incorrectly on multiplicative reasoning to predict far terms, which links to teachers' understanding of patterns and relationships with number. The discussions between the learners during the focus group interviews highlighted their potential to see relations when guided with specific questions. Data recorded in a vertical function table appears to assist in the development of algebraic thinking as it encourages the learners to look across the data. The results show that although the learners demonstrate their potential for functional thinking, the current curricula and textbooks offer little opportunity for this development. Many activities can be adapted and used for this purpose, but the guidance and scaffolding needed by the teacher is not evident in either of the documents. With further support, well designed activities and careful scaffolding, learners' understanding could be improved and their levels of functional thinking could become more sophisticated (Blanton *et al*., 2015).

CHAPTER 5

CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

The previous two chapters addressed the design and methodology of the study as well as the findings and discussion of the analysed data. This chapter revisits the purpose of the research followed by discussion and conclusions from each of the research questions. These are then summarised to highlight important aspects of the research findings related to algebraic thinking in the Foundation Phase. The final section of the chapter focuses on the researcher's reflections, recommendations and suggestions for future research.

The aim of this study was to compare two mathematics curricula to understand how they provide for the development of algebraic thinking in the Foundation Phase.

The main research question is:

> *How do the South African and Singapore Mathematics curricula provide for the development of algebraic thinking in the Foundation Phase?*

5.2 Discussion of research questions

5.2.1 Sub-question 1:

> *How do the CAPS and SM curricula and textbooks provide for the development of algebraic thinking in the Foundation Phase?*

From the findings and analysis from the data related to this question, three issues for discussion have been identified. These issues are (a) recursive versus functional thinking; (b) the importance of activities and (c) instructional sequencing and how to plan for the teaching of functional thinking. These issues will now be discussed further in relation to the research sub-question.

(a) Recursive thinking versus functional thinking

The curriculum documents from both CAPS and SM provide guidelines for teaching mathematics. These detail the expected learning outcomes of a specific subject (Lunenburg, 2011). Remillard (1999) suggests that teachers need clear and well-designed curriculum guidance to explain the content to be covered and to provide clarification in terms of the types of activities to be completed by learners. The content from these documents is then used to write textbooks which the teachers use to select activities for their learners to complete. While the pattern activities currently used in schools are in line with the curriculum and display good linkage between the requirements and the types of activities presented to the learners, many of the tasks remain within the second level, recursive particular, of Blanton *et al*.'s levels of sophistication. This meets the requirements of each curriculum, which focuses on extending and describing patterns. Twohill (2016) suggests that when children are posed with a pattern problem, their natural instinct is to use the recursive approach. Although this is a normal starting point for children, Blanton *et al*. (2015), Moss and London McNab (2011) and Carraher and Schliemann (2016) found that young children have the capacity to think algebraically when given the opportunity to look for functional relationships and describe them to others.

In terms of recursive and functional thinking, the curriculum documents and textbooks favour recursive type patterns and make no mention of functional thinking activities or development thereof. Although the curriculum documents explain the aims and processes of the pattern activities, specific guidance relating to functional thinking and how to move learners from recursive to functional thinking will need to be made more explicit. During their interviews, neither teacher emphasised the idea of functional thinking and the importance of its development. Twohill (2016) explains that the opportunities provided in textbooks and activities should not only favour the recursive approach to patterns. Blanton and Kaput (2011) found that when given ample opportunities to think and work with functions, children start by using their natural language and through carefully planned experiences and scaffolding, can move towards a more symbolic representation where they can develop a generalised rule.

(b) Importance of activities

Mc Auliffe (2013) emphasises the importance of providing opportunities for children to develop their algebraic thinking in the early grades and for this to continue into high school. Some of the activities found in the CAPS and SM textbooks have the potential to develop algebraic thinking, however, adaptations to the curriculum clarification and the activities need to occur in order to develop learners' algebraic thinking. Roberts (2010) found that although examples in a mathematics curriculum could have the potential to develop algebraic thinking, they do not always expect the underlying principles of algebra to be recognised. In both curriculum documents and textbooks used in this study, activities with the potential to develop algebraic thinking were identified. The activity's focus would need to change in order to shift it from an arithmetic task to an algebraic one. Many current textbook activities can be adapted to assist in algebraic thinking development. Activities using vertical function tables are an example of this. A function table activity where the learners must provide the function rule is a useful representation to facilitate the development of algebraic thinking. Another example is found in the CAPS clarification section, which encourages the learners to see the link between counting in 5's and counting in 50's to assist with multiplication. The focus of this type of activity should be to guide the learners towards seeing the relationship between the two patterns and how this relationship could be described in general terms.

There are examples of patterns present in each curricula's textbooks, but there is a lack of evidence in this research of these activities moving learners towards making generalisations. The growing pattern problems could include prediction questions which would make the learner think about the problem differently. Discussions related to predicting a far term begin to build a foundation for further algebraic thinking. Examples where different representations are used could also be adapted.

(c) Instructional sequencing: how to plan for teaching functional thinking

In a study of mathematics textbooks, Vermeulen (2016) found the sequencing of the activities to be an area of concern. Blanton *et al*. (2015) used a particular instructional sequence for selecting functional thinking tasks as used in their research study. They found learners coped better with y= mx type functions and were challenged by functions of the type y= x+b. For this reason, pattern activities related to functions begin with y= mx type functions, then shift to y= a + b and finally to the function type y= mx+b. The tasks selected by Blanton *et al*. (2015) to develop functional thinking were designed to follow a particular instructional sequence: a problem scenario where two quantities co-vary is given; opportunity to organise the data in a table; opportunity to describe any relationships noticed in the data using the table and the opportunity to predict near and further terms. This instructional sequencing of tasks will have an impact on the design of curriculum documents and textbooks. The findings related to this question have demonstrated the need for adaptations to current activities, introduction of more suitable activities and for these activities to be carefully planned in a teaching and learning trajectory that facilitates the development of algebraic thinking.

5.2.2 Sub-question 2

What are the teachers' perceptions of the development of algebraic thinking within each curriculum?

The teacher plays an important role in the shaping of the curriculum and serves as the link between the curriculum and the child's reality (Remillard & Bryans, 2004). The teacher's understanding of algebraic thinking and how this can be developed is vitally important. Greenes (2004) also emphasises the importance of the role of the mathematics teacher in the teaching of early algebra. Two issues related to teachers emerged from the findings and will be discussed in relation to the research sub-question. These are (a) current teacher knowledge of patterns and (b) the need for more specialised knowledge for teaching functional thinking.

(a) Current teacher knowledge of patterns

One of the roles of the teacher is to make the connections between algebra and arithmetic. In order to do this, the teacher requires a fundamental understanding of these connections. Remillard and Bryans (2004) believe that the teacher must engage with and understand the content before she teaches it to her learners. The teachers use their respective curricula to inform their teaching of patterns. They do not, however, talk about patterns in terms of functional thinking where the relationships between terms are the focus. From the interviews, both teachers explained their understanding of patterns as a link to multiplication and multiplicative reasoning and not as a precursor to early algebra. One teacher mentioned, for example, how she makes the link between counting in 3's and counting in 30's explicit. She did not, however, mention anything about describing the relationship between the terms. With the main focus on extending patterns, the responses from both teachers indicate that their understanding of patterns and pattern activities falls within level 2 and 3, recursive particular and recursive general levels of sophistication (Blanton *et al.*, 2015). The teachers' thinking also needs to include functional thinking in order for them to offer these opportunities to their learners. The teachers appear to have a limited concept and understanding of functional thinking and how to move activities out of the arithmetic domain and into an algebraic one. This could occur through professional development with a focus on algebraic thinking.

(b) The need for more specialised knowledge for teaching functional thinking

Ball, Thames and Phelps (2008) researched the knowledge that teachers need in order to teach mathematics. They identified different domains of knowledge they feel are vital for a teacher to possess. Three of the knowledge domains include *common content knowledge (CK), specialised content knowledge (SCK)* and *horizon content knowledge (HK).* In relation to mathematics teaching, common content knowledge refers to the knowledge about maths that most people will have acquired and is not specific only to teachers. Specialised content knowledge is the knowledge needed by the teacher for teaching. Teachers need this type of knowledge for effective instruction in the classroom. An example would be a teacher knowing a variety of methods to solve a problem, while the learners may only use one method. Although this information may never be taught to the learners, the teacher needs it in order to understand errors or misconceptions of learners. Horizon content knowledge is defined by Ball *et al.* (2008:403) as *"*an awareness of how mathematical topics are related over the span of mathematics included in the curriculum*".* This knowledge includes knowing how the current content relates to mathematics learnt previously and to work that is yet to be taught.

The findings of this study suggest that two types of content knowledge could be strengthened in teachers: specialised content knowledge and horizon content knowledge. The current curriculum documents and teacher guides do not seem to offer sufficient pedagogical guidance in terms of using pattern activities to develop functional thinking. International trends, as well as the data from this study, shows that learners at this level have the potential to find relationships between terms and, with scaffolding, can begin to represent these in different ways. One aspect of specialised content knowledge is how to use mathematics activities to their potential. The current models being used in some primary school mathematics, such as tables, stories and function machines have the potential to focus more on algebraic thinking and help children move beyond arithmetic thinking. The questions related to these activities will need to be carefully selected and aim towards finding relationships and predicting far terms. A story asking how many wheels on a specific number of bicycles, for example, remains in the arithmetic domain until the questions relate to predicting a far term, generalising a rule and justifying their thinking. Blanton and Kaput (2011) have found that learners at this age are capable of achieving this. This can, however, only be carried out if the teachers understand how to adapt the activities, utilise spontaneous opportunities for discussions and understand the reasons for the task (Hunter, 2016; Remillard & Bryans, 2004).

5.2.3 Sub-question 3

How do learners using CAPS and SM curricula solve pattern problems?

The data relating to the learners' performance was collected using a pencil and paper test as well as focus group interviews. The results from these collection methods showed a difference in learner performance. The issues emerging from this data are (a) types of activities used in mathematics classrooms and (b) mathematical practices in the classroom.

(a) Types of activities used in mathematics classrooms

The findings from the learners' pencil and paper tests and focus group interviews demonstrate that the majority of the learners are more naturally inclined to work within the recursive domain when solving pattern problems. This result is expected given the types of pattern tasks they are exposed to in class. The pencil and paper test results, in terms of the functional thinking tasks, seemed to imply that the learners were not able to think in a functional way. The function table question, for example, produced low levels of success for both the CAPS and SM groups. The majority of learners remained in the recursive particular level where the emphasis was on counting. Some learners moved into the recursive general level where they were able to conceptualise a general rule, but used it in a recursive way. Their performance in the focus group interviews, demonstrated a different result, however. When completing the dogs and eyes activity during the focus group interviews, which was of a similar function structure ($y=2x$ and $y=3x$) to Question 7 in the pencil and paper test, most of the learners were able to see the relationship between the terms and could explain that the number of eyes is double the number of dogs. The majority of the learners were able to think about the task in a functional way. This could imply a lack of exposure in the classrooms to these types of activities. The curricula may include appropriate activities. The learners are not, however, engaging frequently enough with these types of pattern activities. There is a need for more focus on the teaching and assessment of pattern tasks that require learners to develop functional thinking.

(b) Mathematical practices in the classroom

Exposure to functional thinking tasks create opportunities for learners to engage in the four critical mathematical practices. These mathematical practices include generalising, representing, justifying and reasoning with mathematical structure and relationships (refer to Chapter 2.5). These practices are promoted by Blanton, Brizuela and Stephens (2016) and are crucial in the development of algebraic thinking. During the focus group interviews, where the learners were encouraged to work together, engage in discussions and challenge each other's work and thinking, these mathematical practices were evident and encouraged. Teachers need to organise their classes and activities to include these types of opportunities and the inclusion of the mathematical practices should become part of the mathematics pattern lesson.

5.2.4 Summary of discussion

The findings from this research aimed to understand how two different mathematics curricula provide for the development of algebraic thinking. The CAPS and SM curricula were compared in terms of the types of opportunities they provide in their documents and textbooks, with similar results. The findings show that both curriculum documents define what is required in terms of patterns and algebra. Each curriculum includes a section where the content is explained, serving as a guide for the teachers. The textbook activities relate to the content of their respective curriculum and appear to be a good translation of what is expected, explaining why many of the activities focus on recursive relationships. The discussion of the documents and textbooks highlight the importance of the types of activities done in class and the instructional sequence of these activities The findings of this study demonstrated how many of the textbook activities could be adapted and used in the development of functional thinking (Carraher *et al*., 2006; Hunter, 2016). The findings related to the teachers highlighted the need for teacher professional development relating to Early Algebra and functional thinking. Teachers' specialised content knowledge and horizon content knowledge need to be supported, allowing them to select or design appropriate tasks and identify spontaneous opportunities within the class to develop algebraic thinking and understand how these activities benefit the learners in later years (Hunter, 2016). The learner results demonstrated their potential to think about relationships in a functional way, depending on the manner in which the tasks are presented. The discussions during the focus group interviews again highlighted the importance of carefully selected tasks, correct guidance and scaffolding in ways where the learners have the opportunity to experiment with generalising, representing, justifying and reasoning with mathematical structure and relationships (Pang & Kim, 2018; Blanton, Brizuela & Stephens, 2016).

5.3 Researcher reflections

There are many lessons to learn from engaging in empirical research in terms of theory and practice. Some of the opportunities and limitations of this research are highlighted and discussed to present a critical and reflective summary of the process.

5.3.1 Sample size

The sample consisted of two Grade 3 classes, their class teachers and a focus group from each class. This small sample could be seen as a limitation in that the findings of the study cannot be generalised to a broader community. This was a case study which provided the opportunity to look more deeply at the topic and develop a better understanding of algebraic thinking.

5.3.2 Learner classwork

Access to the learners' classwork books would have provided a better understanding of the types of activities the teachers design for their learners to complete and would have illustrated the teachers' understanding of patterns and pattern activities. These would have allowed me to develop a bigger picture and a deeper understanding of what patterns and algebraic thinking looked like in each group.

5.3.3 Focus group interviews

The testing during the focus group interviews could also be seen as a limitation of the study. The focus group interviews provided opportunities to investigate deeper into how the learners were solving the tasks and what they were thinking during the process. A wider variety of activities including different types of functional thinking problems would have assisted in providing a deeper understanding and possibly different responses.

5.3.4 Theoretical and analytical tool

The levels of sophistication (Blanton *et al*., 2015) acted as both the theoretical and analytical tool in this study. This worked well as the theory used to guide the study and select the examples, and was used again to code the data. This was useful in that often during the study it created a deeper understanding of the framework. In terms of the data analysis process, using this tool to code the data were challenging as the difference between the levels are slight, and often became difficult to distinguish. The researcher tried to alleviate this challenge by using different knowledgeable others to also code the data and compare the results.

5.4 General recommendations and future research

In terms of the overall development of algebraic thinking, this research study reveals that learners in South Africa, irrespective of the mathematics curricula they follow, have the potential to move beyond recursive thinking when solving pattern problems, but are not offered the opportunities to do so. In order to harness this potential, specific guidance and scaffolding activities, for both teachers and learners, are required. Recommendations made from this study are related to ways to provide for the development of algebraic thinking in the Foundation Phase. Three recommendations will be provided and discussed and these are related to (a) curriculum documents and textbooks; (b) teacher professional development and (c) further support for teachers. Suggestions for future research about this topic will also be explained.

5.4.1 Curriculum documents and textbooks

This study has demonstrated the types of activities and tasks available in two different curricula and their respective textbooks. To assist in the development of algebraic thinking, the guidelines of the curricula should be extended and include more detailed explanations as well as more appropriate tasks to deepen the understanding of algebraic thinking. The guidelines should describe what functional thinking is, what it looks like in a Foundation Phase classroom and the role it plays in a child’s mathematical trajectory. The aims of these activities should be made explicit, guiding the teacher to scaffold the learners. Group work should be encouraged and adequately explained where discussions and challenges of each other’s work and thinking should be encouraged and becomes regular classroom practice.

5.4.2 Teacher professional development

A strong recommendation from the findings of this study is the need for professional development of teachers related to Early Algebra and functional thinking in the Foundation Phase. The scaffolding required to move the learners towards functional thinking should start in the early grades and this training would therefore be aimed at Foundation Phase pre and in service teachers (Carpenter *et al*., 2003; Gotze, 2016). Included in this training should be practical ways for teachers to use their current mathematical models and activities for the development of functional thinking. When using existing tasks, teachers need to find the opportunities within the task to develop the child's algebraic thinking, or need to redesign the task if these opportunities do not exist. Training workshops dealing with this should form part of the professional development.

5.4.3 Further support for teachers

Professional development courses and workshops could assist teachers in understanding and incorporating algebraic thinking into their classrooms. Beyond this, additional support should be made available for teachers, especially when adapting or creating appropriate tasks. This support could be in the form of meetings, where a number of schools join together to share ideas and strategies. This community of practice would assist not only in providing ideas of tasks and lessons, but would also provide security for teachers who doubt their abilities.

Future research from this study could include the design and trialling of algebraic thinking activities with Foundation Phase teachers and their experience of using it in their classrooms. A series of functional thinking tasks could be created following the instructional sequence set out by Blanton *et al*. (2015). These could be used in a study where the learners' progress is tracked. The professional development of teachers to understand the importance of algebraic thinking in the Foundation Phase classroom is vital in developing this skill among our learners. The curriculum documents and textbooks guiding teaching and learning need to be explored and adapted, to include tasks and activities that encourage the notion of generalisation and functional relationships.

REFERENCES

Apple, M.W. 1992. The text and cultural politics. *Educational Researcher*, *21*(7): 4-19.

Arends, F., Winnaar, L. & Mosimege, M. 2017. Teacher classroom practices and Mathematics performance in South African schools: A reflection on TIMSS 2011. *South African Journal of Education*, 37(3): 1-11.

Ball, D., Thames, M.H. & Phelps, G. 2008. Content knowledge for teaching: what makes it special? *Journal of Teacher Education,* 59(5): 389 – 407. https://doi.org/10.1177/0022487108324554 [30 October 2018]

Bertram, C. & Christiansen, I. 2014. *Understanding research: an introduction to reading research.* Pretoria: Van Schaik Publishers.

Blanton, M. 2008. *Algebra and the elementary classroom.* Portsmouth: Heinemann.

Blanton, M. & Kaput, J. 2003. Developing elementary teachers: "algebra eyes and ears". *Teaching Children Mathematics, 10* (2). http://www.jstor.org/stable/41198085 [14 July 2017].

Blanton, M.L. & Kaput, J.J. 2004. Functional thinking as a route into algebra in the elementary grades. In Cai, J. & Knuth, E. (eds). *Early Algebraization. Advances in Mathematics Education*. Berlin: Springer: 5-23.

Blanton, M. & Kaput, J. 2011. Functional thinking as a route into algebra in the elementary grades. *International Reviews on Mathematics Education, 37(1): 34-42*.

Blanton, M., Brizuela, B.M., Gardiner, A.M., Sawrey, K. & Newman-Owens, A. 2015. A learning trajectory in 6-year-old's thinking about generalizing functional relationships. *Journal for Research in Mathematics Education,* 46(5):511-558.

Blanton, M., Brizuela, B.M., Gardiner, A.M., Sawrey, K. & Newman-Owens, A. 2015. A learning trajectory in 6-year-ylds' thinking about generalizing functional relationships. *Journal for Research in Mathematics Education,* 46(5): 511-558.

Blanton, M., Brizuela, B.M.& Stephens, A.C. 2016. Elementary children's algebraic thinking. Presented at the *13th International Congress on Mathematical Education (ICME-13).* Hamburg, Germany.

Blanton, M., Brizuela, B.M., Stephens, A., Knuth, E., Isler, I., Gardiner, A.M., Stroud, R., Fonger, N.L. & Stylianou, D. 2018. In Kieran, C. (ed). *Teaching and Learning Algebraic Thinking with 5- to 12- year olds.* Cham: Springer: 27-49.

Britt, M.S. & Irwin, K.C. 2011. Algebraic thinking with and without algebraic representation: a pathway for learning. In Cai, J. & Knuth, E. (eds). *Early Algebraization.* New York: Springer: 137-159.

Cai, N. & Knuth, E.J. 2005. The development of students' algebraic thinking in earlier grades from curricular, instructional and learning perspectives. *Zentralblatt für didaktik der mathematik* (37)1: 1-4.

Carpenter, T., Franke, M. & Levi, L. 2003. *Thinking Mathematically.* Portsmouth: Heinemann.

Carraher, D.W., Schliemann, A.D., Brizuela, B.M. & Earnest, D. 2006. Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education*, (37) 2: 87-115.

Carraher, D.W., Martinez, M.V. & Schliemann, A.D. 2008. Early algebra and mathematical generalisation. *Mathematics Education*, 40 (1): 3-22.

Carraher, D.W. & Schliemann, A.D. 2016. Functional relations in early algebraic thinking. *13th International Congress on Mathematical Education.* July 2016. Hamburg: 24-31.

Cawood, H. 2006. Simply Maths. Westville: Trumpeter Publishers.

Charles, R.I. & Carmel, C.A. 2005. Big ideas and understandings as the foundation for elementary and middle school mathematics. *Journal of Mathematics Education Leadership,* 8(1): 9-24.

Chen, S. 2008. Primary Singapore mathematics, a revolutionary program designed to improve student mathematics achievement: what is Singapore maths? http://www.ccds.org/customized/uploads/files/sing_math_ccds.pdf [9 May 2015]

Cohen, L., Manion, L. & Morrison, K. 2007. *Research methods in education*. 6th ed. New York: Routledge Falmer.

Creswell, J.W. 2003. *Research design: qualitative, quantitative and mixed methods approaches.* Lincoln: SAGE

Day, L. 2017. Generalisation through noticing structure in algebraic reasoning. In Barker, V., Spencer, T. & Manuel, K. (eds). *Capital maths. Proceedings of the 26th Biennial Conference of the Australian Association of Mathematics Teachers Inc, Canberra, 11-13 July 2017.* Adelaide: The University of Notre Dame: 99-107.

Fink, A. 2014. *Conducting research literature reviews: From the internet to paper*. 4th ed. Thousand Oaks, CA: SAGE.

Fong, N.S. 2004. Developing algebraic thinking in early grades: case study of the Singapore primary mathematics curriculum. *The Mathematics Educator,* 8(1): 39-59.

Garner, R. 1992. Learning from school texts. *Educational Psychologist*, *27*(1): 53-63.

Gotze, D. 2016. "It's nothing else than a term"- the epistemic role of language while generalising numerical patterns. Presented at the *13thInternational Congress on Mathematical Education (ICME-13).* Hamburg, Germany.

Greenes, C. 2004. *Algebra: It's elementary*. http://grc.depaul.edu/algebrainitiative/articles/algebra_elementary_article_carole_greene.pdf [5 August 2016].

Haggarty, L. & Pepin, B. 2002. An investigation of mathematics textbooks and their use in English, French and German classrooms: who gets an opportunity to learn what? *British Educational Research Journal*, 28(4): 567-590.

Hamilton, L. & Corbett-Whittier, C. 2013. *Using case study in education research*. London: SAGE.

Hopkins. D. 2014. *A teacher's guide to classroom research.* 5th ed. Berkshire: Open University Press.

Hu, W. 2010. Making math as easy as 1, pause 2, pause. *The New York Times:* 1, October 2010.

Hunter, J. 2016. Scaffolding teacher practice to develop early algebraic reasoning. Presented at the *13th International Congress on Mathematical Education (ICME-13).* Hamburg, Germany.

Johnson, B. & Turner, L.A. 2003. Data Collection Strategies in Mixed Methods Research. In Tashakkori, A. & Teddlie, C. (eds). *Handbook of Mixed Methods in Social and Behavioural Research.* USA: Sage Publications: 297-321.

Kaput, J. 2008. What is algebra? What is algebraic reasoning? In Kaput, J.J., Carraher, D.W. & Blanton, M.L. (eds). *Algebra in the Early Grades.* New York: Lawrence Erlbaum Associates: 5-18.

Kheong, F.G., Ramakrishnan, C. & Choo, M. 2007. *My Pals Are Here Maths Teachers' Planning Guide*. 3rd ed. Singapore: Marshall Cavendish Education.

Kheong, F.G., Ramakrishnan, C. & Choo, M. 2014a. *My Pals Are Here 2A*. Singapore: Marshall Cavendish Education.

Kheong, F.G., Ramakrishnan, C. & Choo, M. 2014b. *My Pals Are Here 2B*. Singapore: Marshall Cavendish Education.

Kieran, C. 2004. Algebraic thinking in the early grades: what is it? *The Mathematics Educator,* (8)1: 139-151.

Kieran, C., Pang, J., Schifter, D. & Ng, S.F. 2016. Early algebra: research into its nature, its learning, its teaching. *ICME 13 Topical surveys.* New York: Springer.

Krajcar, E. 2014. *New wave mental maths*. Cape Town: R.I.C Publications.

Leinwand, S. & Ginsburg, A.L. 2007. Learning from Singapore math. *Educational Leadership: Making Math Count,* 65(3): 32-36.

Lunenburg, F.C. 2011. Theorizing about curriculum: Conceptions and Definitions. *International Journal of Scholarly Academic Intellectual Diversity, 13(1): 1-5.*

Matthews, P., Rittle-Johnson, B., McEldoon, K. & Taylor, R. 2012. Measure for measure: what combining diverse measures reveals about children's understanding of the equal sign as an indicator of mathematical equality. *Journal for Research in Mathematics Education,* 43(3): 316-350.

Maxwell, J.A. 2009. Designing a qualitative study. In Bickman, L. & Rog, D.J. (eds). *The SAGE Handbook of Applied Social Research Methods.* 2nd ed. Los Angeles: SAGE: 214-254.

Maxwell, J.A. 2013. *Qualitative research design: an interactive approach.* 3rd ed. London: SAGE.

Mc Auliffe, S. 2013. The development of preservice teachers' content knowledge for teaching early algebra. Unpublished doctoral dissertation. Cape Peninsula University of Technology, Cape Town.

Mc Auliffe, S. & Lubben, F. 2013. Perspectives on pre-service teacher knowledge for teaching early algebra. *Perspectives in Education: Special Edition,* 31 (3): 155-170.

Morgan, D.L. 2007. Focus groups as qualitative research. http://www.twu.edu/downloads/qualitative-inquiry/Morgan_1997_Focus_groups_as_qualitative_research.pdf [21 January 2016].

Moss, J. & London McNab, S. 2011. An approach to geometric and numeric patterning that fosters second grade students' reasoning and generalizing about functions and co-variation. In Cai, J. & Knuth, E. (eds). *Early Algebraization. Advances in Mathematics Education.* Berlin: Springer.

Mullis, I.V.S., Martin, M.O., Foy, P. & Hooper. M. 2016. *TIMSS 2015 International Mathematics Report*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.

Mulligan, J., Cavanagh, M. & Keanan-Brown, D. 2012. The role of algebra and early algebraic reasoning in the Australian curriculum: mathematics. In Atweh, B., Goos, M., Jorgensen, R. & Siemon, D. (eds). *Engaging the Australian national curriculum: mathematics - perspectives from the field.* St Lucia: Mathematics Education Research Group of Australia: 47-70.

Mwakapenda, W. 2008. Understanding connections in the school mathematics curriculum. *South African Journal of Education,* 28 (2): 189 – 202.

National Research Council. 2001. *Adding it up: helping children learn mathematics.* Kilpatrick, J., Swafford, J. & Findell, B. (eds). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

National Council of Teachers of Mathematics. 2000. *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

Nieuwenhuis, J. 2012. Qualitative research designs and data gathering techniques. In Maree, K. (ed). *First steps in research.* Pretoria: Van Schaik: 70-92.

Olsen, W. 2012. *Data collection: Key debates and methods in social research*. London: SAGE.

Pang, J.S. & Kim, J.W. 2018. Characteristics of Korean students' early algebraic thinking: a generalised arithmetic perspective. In Kieran, C. (ed). *Teaching and learning algebraic thinking with 5- to 12- year olds.* Cham: Springer: 141-165.

Porter, A.C. 2006. Curriculum assessment. In Green, K.L., Camilli, G. & Elmore, P.B. (eds). *Handbook of complementary methods in education research*. Mahwah, NJ: Lawrence Erlbaum: 141- 159.

Powell, T. 2014. A comparative analysis of the Singapore math curriculum and the everyday mathematics curriculum on fifth grade achievement in a large North Eastern urban public school district. Unpublished PhD thesis, Seton Hall University, New Jersey.

Reddy, V., Visser, M., Winnaar, L., Arends, F., Juan, A., Prinsloo, C.H. & Isdale, K. 2016. *TIMSS 2015: Highlights of mathematics and science achievement of Grade 9 South African learners*. Human Sciences Research Council.

Remillard, J.T. 1999. *Curriculum materials in mathematics education reform: a framework for examining teachers' curriculum development.* Philadelphia: University of Pennsylvania.

Remillard, J.T. & Bryans, M.B. 2004. *Teachers' orientations towards mathematics curriculum materials: implications for teacher learning.* Philadelphia: University of Pennsylvania.

Roberts, N. 2010. Comparison at two levels of the content treatment of 'early algebra' in the intended curricula in South Africa and England. In Joubert, M. & Andrews, P. (eds). *Proceedings of the British Congress for Mathematics Education.* University Place, University of Manchester.

Roberts, N. 2012. Patterns, functions and algebra in the South African curriculum: towards more detail for South African teachers. In Nieuwoudt, S., Laubscher, D. & Dreyer, H. (eds.). *Proceedings of the Eighteenth National Congress of the Association for Mathematics Education of South Africa, 24-28 June 2012*. Potchefstroom: North West University: 302 – 319.

Rule, P. & John, V. 2011. *Your guide to case study research.* Pretoria: Van Schaik.

Schmidt, W.H., McKnight, C.C., Valverde, G.A., Houang, R.I. & Wiley, D.E. 1997. *Many visions, many aims. Volume 1. A cross-national investigation of curricular intentions in school mathematics.* London: Kluwer.

Schoenfeld, A. 1987. Cognitive science and mathematics education: an overview. In Schoenfeld, A. (ed). *Cognitive science and mathematics education.* Hillsdale, NJ: Laurence Erlbaum Associates: 1-32.

Siemon, D., Beswick, K., Brady, K., Clark, J., Faragher, R. & Warren, E. 2015. *Teaching mathematics: foundations to middle years.* 2nd ed. Melbourne: Oxford University Press.

Singapore. Ministry of Education. 2012. *Mathematics Syllabus Primary.* https://www.moe.gov.sg/docs/default-source/document/education/syllabuses/sciences/files/primary_mathematics_syllabus_pri1_to_pri5.pdf [2 October 2018].

Sood, S. & Jitendra, A.K. 2007. A Comparative Analysis of Number Sense Instruction in Reform-Based and Traditional Mathematics Textbooks. *Journal of Special Education,* 41(3): 145-157.

South Africa. Department of Basic Education. 2011. *Curriculum and policy assessment statement: Foundation Phase.* Pretoria: Government Printing Works.

South Africa. Department of Basic Education. 2015. *Report on the Annual National Assessments of 2014.* Pretoria: Government Printing Works

South Africa. Department of Basic Education. 2014. Report on the National Senior Certificate examination results. Pretoria: Government Printing Works

Spaull, N. 2013. South Africa's education crisis: the quality of education in South Africa 1994-2011. http://www.section27.org.za/wp-content/uploads/2013/10/Spaull-2013-CDE-report-South-Africas-Education-Crisis.pdf [10 May 2015].

Tarr, J.E., Reys, B.J., Barker, D.D. & Billstein, R. 2006. Selecting high-quality mathematics textbooks. *Mathematics Teaching in the Middle School*, 12(1): 50-54.

Thomas, G. 2011. *How to do your case study: A guide for students and researchers*. London: SAGE.

Twohill, A. 2013. Algebraic reasoning in primary school: developing a framework of growth points. In Smith, C. (ed.) *Proceedings of the British Society for Research into Learning Mathematics, 33(2):* 55-60.

Twohill, A. 2016. The approaches to solution of linear figural patterns adopted by children attending Irish primary schools. Presented at the *13th International Congress on Mathematical Education (ICME-13).* Hamburg, Germany.

Valverde, G. A., & Schmidt, W. H. 1998. Refocusing U.S. math and science education. *Issues in Science and Technology.* 14(2): 60–66.

Venkat, H., & Adler, J. (2012). Coherence and connections in teachers' mathematical discourses in instruction. *Pythagoras*, *33*(3): 1-8.

Vermeulen, N. 2008. *Algebraising the arithmetic curriculum.* Cape Town: Cape Peninsula University of Technology.

Vermeulen, C. 2016. Developing algebraic thinking: the case of South African Grade 4 textbooks. Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education (PME). Szeged: Hungary.

Warren, E. & Cooper T. J. 2008. Generalising the pattern rule for visual growth patterns: actions that support 8 year olds' thinking. *Educational Studies in Mathematics*, 67 (2): 171-185.

Watson, A., Jones, K., & Pratt, D. 2013. *Key ideas in teaching mathematics: Research-based guidance for ages 9–19*. Oxford: Oxford University Press.

Yin, R.K. 2014. *Case study research: design and methods*. 5th ed. Thousand Oaks, CA: SAGE.

Appendix 1: Pencil and paper test

1. 545, _______, 535, 530, 525, _______, ________, ________

   Fill in the missing number. What do you notice?

   ____

   ____

2. 490, 494, _____, 502,______, 510, _______, ________

   Fill in the missing numbers. What do you notice?

   ____

   ____

   What is the same about question 1 and 2?

   ____

   What is different about question 1 and 2?

   ____

3. Fill in the missing numbers:

   a) 4, 8, ___, 16; ___; ___; ___; ___; _____

   b) 40, 80, ___, 160; ___; ___; ___; ___; _____

   What pattern do you notice?

   ____

4. Circle TRUE, FALSE or DON'T KNOW

19 = 9 + 12	TRUE	FALSE	DON'T KNOW
10 + 6 + 5 = 11 + 12	TRUE	FALSE	DON'T KNOW
15 x 4 = 15 + 15 + 15 + 15	TRUE	FALSE	DON'T KNOW
27 = (40 ÷ 2) + (6 + 1)	TRUE	FALSE	DON'T KNOW

5. What symbol would you put in each of the triangles below:

   a) 5 x 6 = 30 then 5 = 30 ∆ 6

   b) 6 + 9 = ∆ + 10

   c) 18 + 9 + 16 = ∆ + 12

   d) ∆ + 7 = 3 + 6 + 4

6. Think of a number, multiply by 2 and add 4. The answer is 30

   a) What is my number? _______

   b) How did you work it out?

   ____

7. Sam grew a sunflower and recorded its growth. On the first day it grew 3 cm, on the second day it was 6cm tall. By the third day it had grown 9cm.

   Complete the table:

Number of days	1	2	3	4	7		15	30
Total number of cm the flower has grown	3	6				30		

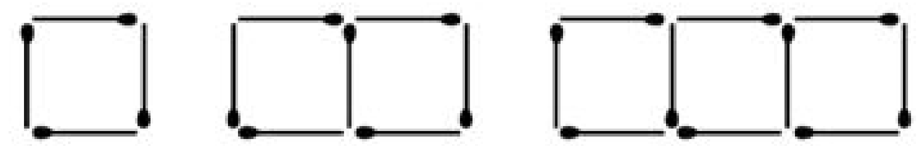
   a) How long will it take for the flower to grow to 150 cm? _______________

   b) How did you work this out?

   ____

   ____

8. Look at the matches pattern below:



Picture 1 Picture 2 Picture 3

a) Can you draw picture 4?

b) Can you draw picture 6?

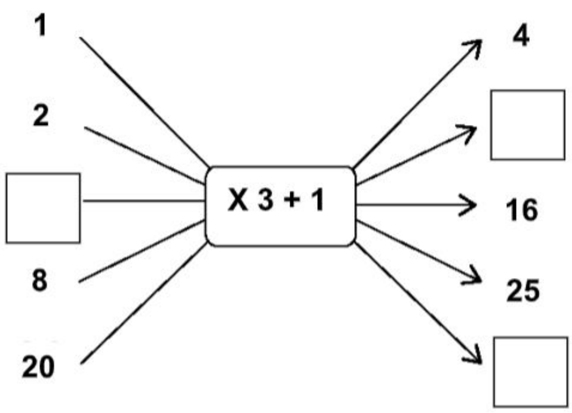
c) How many matches do you need for picture 20? ________________________

d) How did you work this out?

____

____

9. What would fill into the boxes?



Appendix 2: Focus group interview questions

Dogs and eyes:

I went to a park and saw lots of dogs. I wanted to know how many eyes there were altogether. Do you think you could help me work it out?

First I saw 2 dogs, how many eyes?

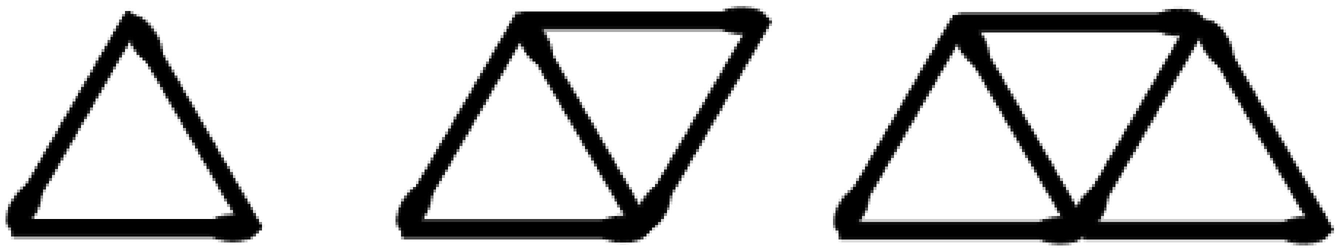
4 dogs?

6 dogs?

Then I saw 10 dogs. How can I work this out? Must I count every dog's eyes? What if there were 20 dogs?

Triangles and sticks:

Look at the triangles we have built using the sticks. With a friend, can you answer the following questions:



How many sticks do we need to make **3** triangles?

How many sticks do we need to make **5** triangles?

How many sticks do we need to make **10** triangles?

What if I wanted to make **20** triangles? Is there a rule we can use?

Appendix 3: Permission letter to parents

Dear Mr/Mrs ……………………………….. 16/11/16

Request for permission for your child to participate in a research study

I am currently conducting research for a Master's Degree in Education at the Cape Peninsula University of Technology (CPUT) in the Faculty of Education and Social Sciences. The focus of my research is: "The development of Algebraic Thinking in the Foundation Phase: A comparative study of two different curricula".

Your child has been selected to participate in a small Mathematics group and her assistance in this project would be greatly appreciated. The group will be given problems to solve and will be asked questions about their workings. The data gathered will assist us in gaining insight to improve programmes and practices in the field of Mathematics.

With your permission, your child will form part of the focus group and will participate in a short Mathematics lesson, during school hours, with other learners from her class. As she works, your child will be interviewed and the lesson will be video recorded. This will assist with the analysis process and will allow me to gain as much information as possible. No names of schools or participants will appear in the research study.

If permission is granted, please sign this letter and return it tomorrow, to confirm your child's participation in this research study. Once completed, the research study will be available for your perusal.

Please feel free to contact me should you require further information.

Yours sincerely, Miss Dominique Afonso

domagafonso@gmail.com 082 506 2679

✂--

I, …………………………………….., give permission for my child, ………………………, to participate in the video recorded Mathematics lesson on Friday, 18 November 2016 at ____________.

Sign:………………………………… Date:………………………………………….

Appendix 4: Letter to teachers

Dear : ___________________

28/11/16

Request for permission to participate in an interview

I am currently conducting research for a Master's Degree in Education at the Cape Peninsula University of Technology (CPUT) in the Faculty of Education and Social Sciences. The focus of my research is: "The development of Algebraic Thinking in the Foundation Phase: A comparative study of two different curricula".

As the class teacher, I would like your permission to interview you where I will enquire about the curriculum you currently teach and the pattern section of this curriculum. The interview will be recorded to assist with the analysis process and will allow me to gain as much information as possible. No names of schools or participants will appear in the research study.

If permission is granted, please complete the reply slip below. Once completed, the research study will be available for your perusal.

Please feel free to contact me should you require further information.

Yours sincerely,

Miss Dominique Afonso

domagafonso@gmail.com

082 506 2679

✂---

I, ……………………………………….., give permission to be interviewed and recorded on 29 November 2016 at ___________________.

Sign:……………………….……… Date: 28/11/2016

Appendix 5: Letter to principal

Cape Peninsula University of Technology

Dear : _________________

Request for permission for your learners to participate in a research study

I am currently conducting research for a Master's Degree in Education at the Cape Peninsula University of Technology (CPUT) in the Faculty of Education and Social Sciences. The focus of my research is: "The development of Algebraic Thinking in the Foundation Phase: A comparative study of two different curricula".

A group of learners from your school have been selected to participate in a small Mathematics group and their assistance in this project would be greatly appreciated. The group will be given problems to solve and will be asked questions about their workings. The data gathered will assist us in gaining insight to improve programmes and practices in the field of Mathematics.

With your permission, your learners will form part of the focus group and will participate in a short Mathematics lesson, during school hours, with other learners from her class. As they work, the learners will be interviewed and the lesson will be video recorded. This will assist with the analysis process and will allow me to gain as much information as possible. No names of schools or participants will appear in the research study.

If permission is granted, please sign the slip below to confirm the participation of your learners in this research study. Once completed, the research study will be available for your perusal.

Please feel free to contact me should you require further information.

Yours sincerely,

Miss Dominique Afonso

domagafonso@gmail.com 082 506 2679

I, ……………………………………., give permission for a group of learners (selected by the class teacher) to participate in the video recorded Mathematics lesson on ……………………..

Sign:……………………………….

Appendix 6: WCED Ethics clearance

Western Cape Government
Education

Audrey.wyngaard@westerncape.gov.za

tel: +27 021 467 9272

Fax: 0865902282

Private Bag x9114, Cape Town, 8000

wced.wcape.gov.za

REFERENCE: 20160824 – 3534

ENQUIRIES: Dr A T Wyngaard

Miss Dominique Afonso

26 Kasteel Street

Bothasig

7441

Dear Miss Dominique Afonso

RESEARCH PROPOSAL: THE DEVELOPMENT OF ALGEBRAIC THINKING IN THE FOUNDATION PHASE: A COMPARATIVE STUDY OF TWO DIFFERENT CURRICULA

Your application to conduct the above-mentioned research in schools in the Western Cape has been approved subject to the following conditions:

1. Principals, educators and learners are under no obligation to assist you in your investigation.
2. Principals, educators, learners and schools should not be identifiable in any way from the results of the investigation.
3. You make all the arrangements concerning your investigation.
4. Educators' programmes are not to be interrupted.
5. The Study is to be conducted from **01 September 2016 till 30 September 2016**
6. No research can be conducted during the fourth term as schools are preparing and finalizing syllabi for examinations (October to December).
7. Should you wish to extend the period of your survey, please contact Dr A.T Wyngaard at the contact numbers above quoting the reference number?
8. A photocopy of this letter is submitted to the principal where the intended research is to be conducted.
9. Your research will be limited to the list of schools as forwarded to the Western Cape Education Department.
10. A brief summary of the content, findings and recommendations is provided to the Director: Research Services.
11. The Department receives a copy of the completed report/dissertation/thesis addressed to:

The Director: Research Services, Western Cape Education Department, Private Bag X9114, CAPE TOWN 8000

We wish you success in your research.Kind regards.

Signed: Dr Audrey T Wyngaard

Directorate: Research DATE: 25 August 201

Appendix 7: CPUT Ethics clearance

Cape Peninsula University of Technology

***For office use only	
Date submitted	13 May 2016
Meeting date	n/a
Approval	Approved
Ethical Clearance number	EFEC 4-6/2016

FACULTY OF EDUCATION

RESEARCH ETHICS CLEARANCE CERTIFICATE

This certificate is issued by the Education Faculty Ethics Committee (EFEC) at Cape Peninsula University of Technology to the applicant/s whose details appear below.

1. Applicant and project details (Applicant to complete this section of the certificate and submit with application as a Word document)

Name(s) of applicant(s):	Dominique Gabriala Afonso
Project/study Title:	Algebraic thinking in the Foundation Phase: a comparative study of two different curricula.
Is this a staff research project, i.e. not for degree purposes?	Yes / **No**
If for degree purposes:	**Degree**: Master's Degree
Funding sources:	**Supervisor(s)**: Dr. Sharon Mc Auliffe

2. Remarks by Education Faculty Ethics Committee:

This Master's research project is granted ethical clearance by the Education Faculty		
Ethics Committee (EFEC) at Cape Peninsula University of Technology. This certificate is		
Valid for **two** calendar years from the date of issue indicated below.		
Approved:	**Referred back:**	**Approved subject to adaptations:**
Chairperson Name: Chiwimbiso Kwenda **Chairperson Signature:**	**Date:** 14 June 2016	
Approval Certificate/Reference: EFEC 4-6/2016		

EFEC Form V6_updated 2016